IIT JAM 2018 QUESTION

Q1. Which one of the following is TRUE?

(a) $\mathbb{Z}_n$ is cyclic if and only if n is prime

(b) Every proper subgroup of $\mathbb{Z}_n$ is cyclic

(c) Every proper subgrop of S_4 is cyclic

(d) If every proper subgroup of a group is cyclic, then the group is cyclic

Q2. Let $a_n = \frac{b_{n+1}}{b_n}$, where $b_1 = 1, b_2 = 1$ and $b_{n+2} = b_n + b_{n+1}, n \in \mathbb{N}$. Then $\lim_{n\to\infty} a_n$ is

(a) $\frac{1-\sqrt{5}}{2}$ (b) $\frac{1-\sqrt{3}}{2}$

(c) $\frac{1+\sqrt{3}}{2}$ (d) $\frac{1+\sqrt{5}}{2}$

Q3. If $\{v_1, v_2, v_3\}$ is a linearly independent set of vectors in a vector space over $\mathbb{R}$, then which one of the following sets is also linearly independent?

(a) $\{v_1 + v_2 - v_3, 2v_1 + v_2 + 3v_3, 5v_1 + 4v_2\}$

(b) $\{v_1 - v_2, v_2 - v_3, v_3 - v_1\}$

(c) $\{v_1 + v_2 - v_3, v_2 + v_3 - v_1, v_3 + v_1 - v_2, v_1 + v_2 + v_3\}$

(d) $\{v_1 + v_2, v_2 + 2v_3, v_3 + 3v_1\}$

Q4. Let a be a positive real number. If f is a continuous and even function defined on the interval $[-a, a]$, then $\int_{-a}^{a} \frac{f(x)}{1+e^x} dx$ is equal to

(a) $\int_0^a f(x)dx$ (b) $2\int_0^a \frac{f(x)}{1+e^x} dx$

(c) $2\int_0^a f(x)dx$ (d) $2a\int_0^a \frac{f(x)}{1+e^x} dx$

Q5. The tangent plane to the surface $z = \sqrt{x^2 + 3y^2}$ at $(1,1,2)$ is given by

(a) $x - 3y + z = 0$ (b) $x + 3y - 2z = 0$

(c) $2x + 4y - 3z = 0$ (d) $3x - 7y + 2z = 0$

Q6. In $\mathbb{R}^3$, the cosine of the acute between the surfaces $x^2 + y^2 + z^2 - 9 = 0$ and $z - x^2 - y^2 + 3 = 0$ at the point $(2,1,2)$ is

(a) $\frac{8}{5\sqrt{21}}$ (b) $\frac{10}{5\sqrt{21}}$

(c) $\frac{8}{3\sqrt{21}}$ (d) $\frac{10}{3\sqrt{21}}$

Q7. Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a scalar field, $\vec{v}: \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field and let $\vec{a} \in \mathbb{R}^3$ be a constant vector. If $\vec{r}$ represent the position vector $x\hat{i} + y\hat{j} + z\hat{k}$, then which one of the following is FALSE?

(a) $curl(f\vec{v}) = grad(f) \times \vec{v} + f\ curl(\vec{v})$

(b) $div(grad(f)) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$

(c) $curl(\vec{a} \times \vec{r}) = 2|\vec{a}|\vec{r}$

(d) $div\left(\frac{\vec{r}}{|\vec{r}|^3}\right) = 0$, for $\vec{r} \neq \vec{0}$

Q8. In $\mathbb{R}^2$, the family of trajectories orthogonal to the family of asteroids $x^{2/3} + y^{2/3} = a^{2/3}$ is given by

(a) $x^{4/3} + y^{4/3} = c^{4/3}$ (b) $x^{4/3} - y^{4/3} = c^{4/3}$

(c) $x^{5/3} - y^{5/3} = c^{5/3}$ (d) $x^{2/3} - y^{2/3} = c^{2/3}$

Q9. Consider the vector space V over $\mathbb{R}$ of polynomial functions of degree less than or equal to 3 defined on $\mathbb{R}$. Let $T: V \to V$ be defined by $(Tf)(x) = f(x) - xf'(x)$. Then the rank of T is

(a) 1 (b) 2
(c) 3 (d) 4

Q10. Let $S_n = 1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}$ for $n\in\mathbb{N}$. Then which one of the following is TRUE for the sequence $\{s_n\}_{n=1}^{\infty}$

(a) $\{s_n\}_{n=1}^{\infty}$ converges in $\mathbb{Q}$

(b) $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence but does not converge in $\mathbb{Q}$

(c) the subsequence $\{S_{k^n}\}_{n=1}^{\infty}$ is convergent in $\mathbb{R}$, only when k is even natural number

(d) $\{S_n\}_{n=1}^{\infty}$ is not a Cauchy sequence

Q11. Let $a_n = \begin{cases} 2+\frac{(-1)^{\frac{n-1}{2}}}{n}, & if\ n\ is\ odd \\ 1+\frac{1}{2^n}, & if\ n\ is\ even \end{cases}, n\in\mathbb{N}.$

Then which one of the following is TRUE?

(a) $\sup\{a_n \mid n\in\mathbb{N}\}=3$ and $\inf\{a_n \mid n\in\mathbb{N}\}=1$

(b) $\liminf (a_n)=\limsup(a_n)=\frac{3}{2}$

(c) $\sup\{a_n \mid n\in\mathbb{N}\}=2$ and $\inf\{a_n \mid n\in\mathbb{N}\}=1$

(d) $\liminf (a_n)=1$ and $\limsup(a_n)=3$

Q12. Let $a,b,c\in\mathbb{R}$. Which of the following values of a,b,c do NOT result in the convergence of the series $\sum_{n=3}^{\infty}\frac{a^n}{n^b(\log_e n)^c}$?

(a) $|a|<1, b\in\mathbb{R}, c\in\mathbb{R}$

(b) $a=1, b>1, c\in\mathbb{R}$

(c) $a=1, b\geq 0, c<1$

(d) $a=-1, b\geq 0, c>0$

Q13. Let $a_n = n+\frac{1}{n}, n\in\mathbb{N}$. Then the sum of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{a_{n+1}}{n!}$ is

(a) $e^{-1}-1$ (b) e^{-1}

(c) $1-e^{-1}$ (d) $1+e^{-1}$

Q14. Let $a_n = \frac{(-1)^n}{\sqrt{1+n}}$ and $c_n=\sum_{k=0}^{n}a_{n-k}a_k$, where $n\in\mathbb{N}\cup\{0\}$. Then which one of the following is TRUE?

(a) Both $\sum_{n=0}^{\infty}a_n$ and $\sum_{n=1}^{\infty}c_n$ are convergent

(b) $\sum_{n=1}^{\infty}a_n$ is convergent but $\sum_{n=1}^{\infty}c_n$ is not convergent

(c) $\sum_{n=1}^{\infty}c_n$ is convergent but $\sum_{n=0}^{\infty}a_n$ is not convergent

(d) Niether $\sum_{n=0}^{\infty}a_n$ nor $\sum_{n=1}^{\infty}c_n$ is convergent

Q15. Suppose that $f,g:\mathbb{R}\to\mathbb{R}$ are differentiable functions such that f is strictly increasing and g is strictly decreasing. Define $p(x)=f(g(x))$ and $q(x)g(f(x)), \forall x\in\mathbb{R}$. Then, for $t>0$, sign of $\int_0^t p'(x)(q'(x)-3)dx$ is

(a) positive (b) negative
(c) dependent on t (d) dependenton f and g

Q16. For $x\in\mathbb{R}$, let $f(x)=\begin{cases} x^3\sin\left(\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0\end{cases}$. Then which one of the following is FALSE?

(a) $\lim_{x\to 0}\frac{f(x)}{x}=0$

(b) $\lim_{x\to 0}\frac{f(x)}{x^2}=0$

(c) $\frac{f(x)}{x^2}$ has infinitely many maxima and minima on the interval (0,1)

(d) $\frac{f(x)}{x^4}$ is continuous at $x=0$ but not differentiable at $x=0$

Q17. Let

$$f(x,y)=\begin{cases}\frac{xy}{(x^2+y^2)^{\alpha}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Then which one of the following is TRUE for f at the point (0,0)?

(a) For $\alpha = 1, f$ is continuous but not differentiable

(b) For $\alpha = \frac{1}{2}, f$ is continuous and differentiable

(c) For $\alpha = \frac{1}{4}, f$ is continuous and differentiable

(d) For $\alpha = \frac{3}{4}, f$ is neither continuous nor differentiable.

Q18. Let $a, b \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function. If $z = e^u f(v)$, where $u = ax + by$ and $v = ax - by$, then which one of the following is TRUE?

(a) $b^2 z_{xx} - a^2 z_{yy} = 4a^2 b^2 e^u f'(v)$

(b) $b^2 z_{xx} - a^2 z_{yy} = -4e^u f'(v)$

(c) $bz_x + az_y = abz$

(d) $bz_x + az_y = -abz$

Q19. Consider the region D in the yz plane bounded by the line $y = \frac{1}{2}$ and the curve $y^2 + z^2 = 1$, where $y \geq 0$. If the region D is revolved about the z - axis in $\mathbb{R}^3$, then the volume of the resulting solid is

(a) $\frac{\pi}{\sqrt{3}}$ (b) $\frac{2\pi}{\sqrt{3}}$

(c) $\frac{\pi\sqrt{3}}{2}$ (d) $\pi\sqrt{3}$

Q20. If $\vec{F}(x, y) = (3x - 8y)\hat{i} + (4y - 6xy)\hat{j}$" for $(x, y) \in \mathbb{R}^2$, $\oint_C \vec{F}.d\vec{r}$, where C is the boundary of the triagular region bounded by the lines $x = 0, y = 0$ and $x + y = 1$ oriented in the anti-clockwise direction, is

(a) $\frac{5}{2}$ (b) 3

(c) 4 (d) 5

Q21. Let U, V and W be finite dimensional real vector spaces, $T : U \to V$, $S : V \to W$ and $P : W \to U$ be linear transformations. If range(ST)=nullspace(P),nullspace(ST)=range(P) nad rank(T)=rank(S), then which one fo the following is TRUE?

(a) nullity of T = nullity of S

(b) dimension of U $\neq$ dimension of W

(c) If dimesion of V = 3, dimension of U=4, then P is not identically zero

(d) If dimension of V = 4, dimensio of U = 3 and T is one-one, then P is identically zero

Q22. Let $y(x)$ be the solution of the differential equation $\frac{dy}{dx} + y = f(x)$, for $x \geq 0, y(0) = 0$, where

$f(x) = \begin{cases} 2, & 0 \leq x < 1 \\ 0, & x \geq \end{cases}$. Then $y(x) =$

(a) $2(1 - e^{-x})$ when $0 \leq x < 1$ and $2(e - 1)e^{-x}$ when $x \geq 1$

(b) $2(1 - e^{-x})$ when $0 \leq x < 1$ and 0 when $x \geq 1$

(c) $2(1 - e^{-x})$ when $0 \leq x < 1$ and $2(1 - e^{-1})e^{-x}$ when $x \geq 1$

(d) $2(1 - e^{-x})$ when $0 \leq x < 1$ and $2e^{1-x}$ when $2e^{1-x}$

Q23. An integrating factor of the differential equation $\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right) dx + \frac{1}{4}(x + xy^2)dy = 0$ is

(a) x^2 (b) $3\log_e x$

(c) x^3 (d) $2\log_e x$

Q24. A particular integral of the differential equation $y'' + 3y' + 2y = e^{e^x}$ is

(a) $e^{e^x} e^{-x}$ (b) $e^{e^x} e^{-2x}$

(c) $e^{e^x} e^{2x}$ (d) $e^{e^x} e^{x}$

Q25. Let G be a group satisfying the property that $f : G \to \mathbb{Z}_{221}$ is a homomorphism implies $f(g) = 0, \forall g \in G$. Then a possible group G is

(a) $\mathbb{Z}_{21}$ (b) $\mathbb{Z}_{51}$

(c) $\mathbb{Z}_{91}$ (d) $\mathbb{Z}_{119}$

Q26. Let H be the quotient group $\mathbb{Q}/\mathbb{Z}$. Consider the following statements.

I. Every cyclic subgroup of H is finite

II. Every finite cyclic group is isomorphic to a subgroup of H.

Which one of the following holds?

Q27. Let I denote the 4×4 idenity matrix. If the roots of the characteristic polynomial of a 4×4 matrix M are $\pm\sqrt{\frac{1\pm\sqrt{5}}{2}}$, then $M^8 =$

(a) $I+M^2$ (b) $2I+M^2$
(c) $2I+3M^2$ (d) $3I+2M^2$

Q28. Consider the group $\mathbb{Z}^2=\{(a,b)\mid a,b\in\mathbb{Z}\}$ under component-wise addition. Then which of the following is a subgroup of $\mathbb{Z}^2$?

(a) $\{(a,b)\in\mathbb{Z}^2 \mid ab=0\}$
(b) $\{(a,b)\in\mathbb{Z}^2 \mid 3a+2b=15\}$
(c) $\{(a,b)\in\mathbb{Z}^2 \mid 7 \text{ divides } ab\}$
(d) $\{(a,b)\in\mathbb{Z}^2 \mid 2$ divides a and 3 divides b$\}$

Q29. Let $f:\mathbb{R}\to\mathbb{R}$ be a function and let J be a bounded open interval in $\mathbb{R}$. Define $W(f,J)=\sup\{f(x)\mid x\in J\}-\inf\{f(x)\mid x\in J\}$. Which one of the following is FALSE?

(a) $W(f,J_1)\le W(f,J_2)$ if $J_1\subset J_2$
(b) If f is a bounded function in J and $J\supset J_1\supset J_2 \ldots\supset J_n\supset\ldots$ such that the length of the interval J_n tends to 0 as $n\to\infty$, then $\lim_{n\to\infty} W(f,J_n)=0$
(D) If f is continuous at a point $a\in J$, then for any given $\in>0$ there exists an interval $I\subset J$ such that $W(f,I)<\in$

Q30. For $x>\frac{-1}{2}$, let

$f_1(x)=\frac{2x}{1+2x}, f_2(x)=\log_e(1+2x)$ and $f_3(x)=2x$. Then which one of the following is TRUE?

(a) $f_3(x)<f_2(x)<f_1(x)$ for $0<x<\frac{\sqrt{3}}{2}$
(b) $f_1(x)<f_3(x)<f_2(x)$ for $x>0$
(c) $f_1(x)+f_2(x)<\frac{f_3(x)}{2}$ for $x>\frac{\sqrt{3}}{2}$
(d) $f_2(x)<f_1(x)<f_3(x)$ for $x>0$

Q31-Q.40 carry two marks each.

Q31. Let $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be define by $f(x)=x+\frac{1}{x^3}$.

On which of the following interval(s) is f one-one?

(a) $(-\infty,-1)$ (b) $(0,1)$
(c) $(0,2)$ (d) $(0,\infty)$

Q32. The solution(s) of the differential equation $\frac{dy}{dx}=(\sin 2x)y^{1/3}$ satisfying $y(0)=0$ is (are)

(a) $y(x)=0$ (b) $y(x)=-\sqrt{\frac{8}{27}}\sin^3 x$
(c) $y(x)=\sqrt{\frac{8}{27}}\sin^3 x$ (d) $y(x)=\sqrt{\frac{8}{27}}\cos^3 x$

Q33. Suppose f,g,h are permutations of the set $(\alpha,\beta,\gamma,\delta)$, where

f interchanges α and β but fixes γ and δ,
g interchanges β and γ but fixes α and δ,
h interchanges γ and δ, but fixes α and β.

Which of the following permutations intechange(s) α and δ but fix(es) β and γ?

(a) $f\circ g\circ h\circ g\circ f$ (b) $g\circ h\circ f\circ h\circ g$
(c) $g\circ f\circ h\circ f\circ g$ (d) $h\circ g\circ f\circ g\circ h$

Q34. Let P and Q be two non-empty disjoint subsets of $\mathbb{R}$. Which of the following is (are) FALSE?

(a) If P and Q are compact, then $P\cup Q$ is also compact
(b) If P and Q are not connected, then $P\cup Q$ is also not connected
(c) If $P\cup Q$ and not closed, then Q is closed
(d) If $P\cup Q$ and P are ope, then Q is open

Q35. Let $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ denote the group of non-zero complex numbers multiplication. Suppose $Y_n=\{z\in\mathbb{C}\mid z^n=1\}, n\in\mathbb{N}$. Which of the following is (are) subgroup(s) of $\mathbb{C}^*$?

(a) $\bigcup_{n=1}^{100} Y_n$ (b) $\bigcup_{n=1}^{\infty} Y_{2^n}$

(c) $\bigcup_{n=100}^{\infty} Y_n$ (d) $\bigcup_{n=1}^{\infty} Y_n$

Q36. Suppose $\alpha, \beta, \gamma \in \mathbb{R}$. Consider the following system of linear equations.
$x+y+z=\alpha, x+\beta y+z=\gamma, x+y+\alpha z=\beta$. I f this system has at least solution, then which of the following statements is (are) TRUE?

(a) If $\alpha=1$ then $\gamma=1$ (b) If $\beta=1$ then $\gamma=\alpha$

(c) If $\beta \neq 1$ then $\alpha=1$ (d) If $\gamma=1$ then $\alpha=1$

Q37. Let $\gamma=1m, n \in \mathbb{N},\ m<n,\ P \in M_{n\times m}(\mathbb{R})$, $Q \in M_{m\times n}(\mathbb{R})$. Then which of the following is (are) NOT possible?

(a) $rank(PQ)=n$

(b) $rank(QP)=m$

(c) $rank(PQ)=m$

(d) $rank(QP)=\left\lceil \frac{m+n}{2} \right\rceil$, the smallest integer than or equal to $\frac{m+n}{2}$

Q38. I f $\vec{F}(x,y,z)=(2x+3yz)\hat{i}+(3xz+2y)\hat{j}+(3xy+2z)\hat{k}$ for $(x,y,z)\in\mathbb{R}^3$, then which among the following is (are) TRUE?

(a) $\nabla\times\vec{F}=\vec{0}$

(b) $\oint_C \vec{F}\cdot d\vec{r}=0$ along any simple closed curve C

(c) There exists a scalar function $\phi:\mathbb{R}^3\to\mathbb{R}$ such that $\nabla\cdot\vec{F}=\phi_{xx}+\phi_{yy}+\phi_{zz}$

(d) $\nabla\cdot\vec{F}=0$

Q39. Which of the following subsets of $\mathbb{R}$ is (are) connected?

(a) $\{x\in\mathbb{R} \mid x^2+x>4\}$

(b) $\{x\in\mathbb{R} \mid x^2+x<4\}$

(c) $\{x\in\mathbb{R} \mid |x|<|x-4|\}$

(d) $\{x\in\mathbb{R} \mid |x|>|x-4|\}$

Q40. Let S be a subset of $\mathbb{R}$ such that 2018 is an interior point of S. Which of the following is (are) TRUE?

(a) S contains an interval

(b) There is an seqeunce is S which does not converge to 2018

(c) There is an element $y\in S,\ y\neq 2018$ such that y is also an interior point of S

(d) There is a point $z\in S$, such that $|z-2018|=0.002018$

Q41.-Q.50 Carry one mark each.

Q41. The order of the element (1 2 3) (2 4 5) (4 5 6) in the group S_6 is _________

Q42. Let $\phi(x,y,z)=3y^2+3yz$ for $(x,y,z)\in\mathbb{R}^3$. Then the absolute value of the directional derivative of ϕ in the direction of the line $\frac{x-1}{2}=\frac{y-2}{-1}=\frac{z}{-2}$, at the point $(1,-2,1)$ is _________

Q43. Let $f(x)=\sum_{n=0}^{\infty}(-1)^n x(x-1)^n$ for $0<x<2$. Then the value of $f\left(\frac{\pi}{4}\right)$ is __________

Q44. Let $f:\mathbb{R}^2\to\mathbb{R}$ be given by

$$f(x,y)=\begin{cases}\dfrac{x^2y(x-y)}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Then $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)-\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ at the point (0,0) is ___________

Q45. L e t $f(x,y)=\sqrt{x^3y}\sin\left(\frac{\pi}{2}e^{\left(\frac{y}{x}-1\right)}\right)+xy\cos\left(\frac{\pi}{3}e^{\left(\frac{x}{y}-1\right)}\right)$ for $(x,y)\in\mathbb{R}^2, x>0, y>0$. Then $f_x(1,1)+f_y(1,1)=$ ___________

Q46. Let $f:[0,\infty)\to[0,\infty)$ be continuous on $[0,\infty)$ and differentiable on $[0,\infty)$ .If $f(x)=\int_0^x\sqrt{f(t)}dt$, then $f(6)=$ _________

Q47. Let $a_n=\frac{(1+(-1)^n)}{2^n}+\frac{(1+(-1)^{n-1})}{3^n}$. Then the radius of convergence of the power series

$\sum_{n=1}^{\infty} a_n x^n$ about $x=0$ is ________

Q48. Let A_6 be the group of even permutations of 6 distinct symbols. Then the number of elements of order 6 in A_6 is __________

Q49. Let W_1 be the real vector space of all 5×2 matrices such that the sum of the entries in each row is zero. Let W_2 be the real vector space of all 5×2 matrices such that the sum of the entries in each column is zero. Then the dimension of the space $W_1 \cap W_2$ is _____

Q50. The coefficient of x^4 in the power series expansion of $e^{\sin x}$ about $x=0$ is __________(correct up to three decimal places).

Q51-Q.60 Carry two marks each.

Q51. Let $a_k = (-1)^{k-1}, s_n = a_1 + a_2 + \ldots + a_n$ and $\sigma_n = (s_1 + s_2 + \ldots + s_n)/n$, where $k, n \in \mathbb{N}$. Then $\lim_{n\to\infty} \sigma_n$ is _________(correct up to one decimal place).

Q52. Let $f: \mathbb{R} \to \mathbb{R}$ such that f'' is continuous on $\mathbb{R}$ and $f(0)=1, f'(0)=0$ and $f''(0)=-1$. Then $\lim_{x\to\infty}\left(f\left(\frac{2}{x}\right)\right)^x$ is ___________(correct up to three decimal places).

Q53. Suppose x, y, z are positive real numbers such that $x+2y+3z=1$. If M is the maximum value of xyz^2, then the value of $\frac{1}{M}$ is ____________

Q54. If the volume of the solid in $\mathbb{R}^3$ bounded by the surfaces

$x=-1,\ x=1,\ y=-1,\ y=1,\ z=2,\ y^2+z^2=2$

is $\alpha - \pi$, then $\alpha =$ ___________

Q55. If $\alpha = \int_{\pi/6}^{\pi/3} \frac{\sin t + \cos t}{\sqrt{\sin 2t}}\, dt$, then the value of $\left(2\sin\frac{\alpha}{2}+1\right)^2$ is

Q56. The value of the integral $\int_0^1 \int_x^1 y^4 e^{xy^2}\, dy dx$ is _________(correct up to three decimal places).

Q57. Suppose $Q \in M_{3\times 3}(\mathbb{R})$ is a matrix of rank 2. Let $T: M_{3\times 3}(\mathbb{R}) \to M_{3\times 3}(\mathbb{R})$ be the linear transformation defined by $T(P) = QP$. Then the rank of T is ___________

Q58. The area of the parametrized surface $S = \{((2+\cos u)\cos v, (2+\cos u)\sin v, \sin u) \in \mathbb{R}^3 \mid 0 \le u \le \frac{\pi}{2}, 0 \le v \le \frac{\pi}{2}\}$ is __________(correct up to two decimal places).

Q59. If $x(t)$ is the solution to the differential equation $\frac{dx}{dt} = x^2 t^3 + xt$, for $t > 0$, satisfying $x(0)=1$, then the value of $x(\sqrt{2})$ is ________(correct up to two decimal places).

Q60. If $y(x) = v(x)\sec x$ is the solution of $y'' - (2\tan x)y' + 5y = 0, -\frac{\pi}{2} < x < \frac{\pi}{2}$, satisfying $y(0)=0$ and $y'(0)=\sqrt{6}$, then $v\left(\frac{\pi}{6\sqrt{6}}\right)$ is _________(correct up two decimal places).

IIT JAM 2018 SOLUTION

Q1. (B)

Sol: Every proper subgroup of z_n is cyclic

Q2. (D)

Sol.: $b_{n+2} = b_n + b_{n+1}$

$\Rightarrow \quad \dfrac{b_{n+2}}{b_{n+1}} = \dfrac{b_n}{b_{n+1}} + 1 \qquad \ldots(i)$

Let $\underset{n\to\infty}{lt} \dfrac{b_{n+2}}{b_{n+1}} a_n = \underset{n\to\infty}{lt} \dfrac{b_{n+1}}{b_n} = l$

So, $\underset{n\to\infty}{lt} \dfrac{b_{n+2}}{b_{n+1}} = \underset{n\to\infty}{lt} \dfrac{1}{\left(\dfrac{b_{n+1}}{b_n}\right)} + 1$

$\Rightarrow \quad l = \dfrac{1}{l} + 1 \Rightarrow l^2 - l - 1 = 0$

$\Rightarrow \quad l = \dfrac{1 \pm \sqrt{5}}{2}$; As $l > 1$, so $l = \dfrac{1+\sqrt{5}}{2}$

Q3. (D)

Sol.: For the set

$S = \{v_1 + v_2, v_2 + 2v_3, v_3 + 3v_1\}$

coefficient matrix is

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix}$ whose determinant is 7 (i.e., non-zero) so matrix is of rank 3. Hence S is linearly independent.

Q4. (A)

Sol.: $I = \int_{-a}^{a} \dfrac{f(x)}{1+e^x} dx = \int_{-a}^{a} \dfrac{f(-x)}{1+e^{-x}} dx$

$= \int_{-a}^{a} \dfrac{f(x)}{1+e^{-x}} dx$ (as $f(x)$ is even function)

$= \int_{-a}^{a} \dfrac{e^x f(x)}{e^x + 1} dx$

$\Rightarrow 2I = \int_{-a}^{a} \dfrac{(1+e^x) f(x)}{(1+e^x)} dx = \int_{-a}^{a} f(x) dx$

$= 2\int_0^a f(x) dx \Rightarrow I = \int_0^1 f(x) dx$

Q5. (B)

Sol.: $z = \sqrt{x^2 + 3y^2} \Rightarrow z_x = \dfrac{2x}{2\sqrt{x^2+3y^2}}$ &

$z_y = \dfrac{6y}{2\sqrt{x^2+3y^2}}$. At (1,1,2) we have

$z_x = \dfrac{1}{2}$ & $z_y = \dfrac{3}{2}$

The tangent plane to the surface at (1,1,2) is given by

$(x-1)z_x + (y-1)z_y + (z-2)(-1) = 0$

$\Rightarrow \quad \dfrac{1}{2}(x-1) + \dfrac{3}{2}(y-1) - (z-2) = 0$

$\Rightarrow \quad x + 3y - 2z = 0$.

Q6. (C)

Sol.: Normal Vectors to the surfaces

$x^2 + y^2 + z^2 - 9 = 0$ and $z - x^2 - y^2 + 3 = 0$

at the point (2, 1, 2) is

$4\hat{i} + 2\hat{j} + 4\hat{k}$ & $-4\hat{i} - 2\hat{j} + \hat{k}$

So cosine of acute angle between them will be

$\left| \dfrac{4(-1) + 2(-2) + 4(1)}{\sqrt{(4)^2 + (2)^2 + (4)^2}\sqrt{(-4)^2 + (-2)^2 + (1)^2}} \right|$

$= \dfrac{16}{6\sqrt{21}} = \dfrac{8}{3\sqrt{21}}$

Q7. (C)

Sol.: Let $\vec{a} = a_1 i + a_2 j + a_3 k$ then

$\text{curl}(\vec{a} \times \vec{r}) = \nabla \times (\vec{a} \times \vec{r})$

$$\vec{a}\times\vec{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= (a_2 z - a_3 y)i - (a_1 z - a_3 x)j + (a_1 y - a_2 x)k$$

$$\nabla\times(\vec{a}\times\vec{r}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & (-a_1 z + a_3 x) & a_1 y - a_2 x \end{vmatrix}$$

$$= 2(a_1 + a_2 + a_3) \neq 2|\vec{a}|\vec{r}$$

Q8. (B)

Sol.: $x^{2/3} + y^{2/3} = a^{2/3}$

$\Rightarrow \quad \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$

It's orthogonal trajectory will be given by differential equation

$x^{-1/3} + y^{-1/3}\left(-\frac{dx}{dy}\right) = 0$

$\Rightarrow \quad x^{-1/3} = y^{-1/3}\frac{dy}{dy}$

$\Rightarrow \quad x^{1/3}dx - y^{1/3}dy = 0$

$\Rightarrow \quad x^{4/3} - y^{4/3} = c^{4/3}$

Q9. (C)

Sol.: $T : P_3(R) \to P_3(R)$ is given by

$(Tf)(x) = f(x) - xf'(x)$

$\Rightarrow \quad T(a + bx + cx^2 + dx^3) =$

$(a + bx + cx^2 + dx^3) - x(b + 2cx + 3dx^2)$

$= a - cx^2 - 2dx^3$

$\Rightarrow \quad R(T) = L(\{1, x^2, x^3\})$ so

Rank of $T = 3$.

Q10. (B)

Sol.: $\{s_n\}_{n=1}^{\infty}; s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$

converges in R to e, so it is Cauchy sequence, but $e \notin Q$ so it does not converge in Q.

Q11. (A)

Sol.: $a_1 = 3 = \sup\{a_n \mid n \in \mathbb{N}\}$ &

$\lim_{n\to\infty} a_{2n} = 1 = \inf\{a_n \mid n \in \mathbb{N}\}$

Q12. (C)

Sol.: At $a = 1, b = \frac{1}{2}; c = 0$ series is

$\sum_{n=3}^{\infty}\frac{1}{n^{1/2}}$ which is divergent

Q13. (D)

Sol.: $a_n = n + \frac{1}{n}$ & $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{a_{n+1}}{n!}$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{(n+1) + \frac{1}{(n+1)}}{n!}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\cdot\left(\frac{1}{(n-1)!} + \frac{1}{n!} + \frac{1}{(n+1)!}\right)$$

$$= e^{-1} + 1 - e^{-1} + e^{-1} - (1 - 1) = 1 + e^{-1}.$$

Q14. (B)

Sol.: $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{1+n}}$ is convergent by Leibnitz test

$$c_n = \sum_{k=0}^{n} a_{n-k}a_k = \sum_{k=0}^{n}\frac{(-1)^{n-k}}{\sqrt{1+n-k}}\times\frac{(-1)^k}{\sqrt{1+k}}$$

$$= \sum_{k\to\infty}^{n}\frac{(-1)^n}{\sqrt{(1+k)(1+n-k)}}$$

Now $\lim_{n\to h} c_n \neq 0$, so $\sum_{n=1}^{\infty} c_n$ is not convergent

Q15. (A)

Sol.: $p(x) = f(g(x)) \Rightarrow p'(x) = (f'(g(x))(g'(x))$

$q(x) = g(f(x)) \Rightarrow q'(x) = (g'(f(x))(f'(x))$

as f' is always positive and g' is always negative, so

$p'(x) < 0$ & $q'(x) < 0 \quad \forall x \in R.$

so, $p'(x)(q'(x) - 3) > 0 \quad \forall x.$

So, $\forall t > 0, \int_0^t p'(x)(q'(x) - 3)dx > 0.$

Hence it's sign is always positive

Q16. (D)

Sol.: $\lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$ (True)

$\lim_{x\to 0}\frac{f(x)}{x^2} = \lim_{x\to 0} x\sin\frac{1}{x} = 0$ (True)

Also $\frac{f(x)}{x^2} = x\sin\frac{1}{x}$ in $(0,1)$ has infinitely maxima and minima on the interval $(0,1)$. (True)

$\underset{x\to 0}{lt}\frac{f(x)}{x^4} = \underset{x\to 0}{lt}\frac{1}{x}\sin\left(\frac{1}{x}\right)$ does not exist (False)

Q17. (C)

Sol.: For $\alpha = \frac{1}{4}$;

$$f(x,y) = \begin{cases} \frac{xy}{(x^2+y^2)^{1/4}}; & (x,y)\neq(0,0) \\ 0, \quad ; & (x,y)=(0,0) \end{cases}$$

$$\underset{(x,y)\to(0,0)}{lt} f(x,y) = \underset{r\to 0}{lt}\frac{r^2\cos\theta\sin\theta}{r^{1/2}} = 0$$

Also, $f_x(0,0)=0$; $f_y(0,0)=0$

$\Rightarrow \quad f(h,k)-f(0,0) = 0.h + 0.k + \sqrt{h^2+k^2}\,g(h,k)$

$\Rightarrow \quad g(h,k) = \frac{hk}{(h^2+k^2)^{3/4}}$

Now $\lim_{(h,k)\to(0,0)} g(h,k) = \lim_{r\to 0}\frac{r^2\cos\theta\sin\theta}{r^{3/2}}$

$= \lim_{r\to 0} r^{1/2}\cos\theta\sin\theta = 0.$

So, $f(x,y)$ is both continuous and differentiable at (0,0)

Q18. (A)

Sol.: $z = e^u f(v) = e^{ax+by} f(ax-by)$

$\Rightarrow \quad z_x = ae^{ax+by} f(ax-by) + ae^{ax+by} f'(ax-by)$

$\Rightarrow \quad z_{xx} = a^2 e^{ax+by}\left[f(v)+2f'(v)+f''(v)\right]$

$= a^2 e^u (f(v)+2f'(v)+f''(v)) \quad \ldots$(i)

Similarly

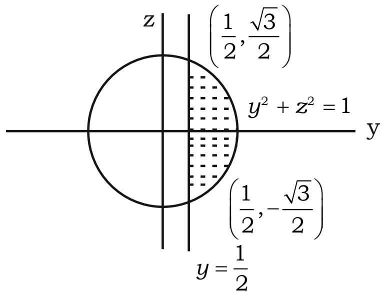
$z_{yy} = b^2 e^u (f(v)-2f'(v)+f''(v)) \quad \ldots$(ii)

From (i) & (ii) we get

$b^2 z_{xx} - a^2 z_{yy} = 4a^2b^2e^u f'(v).$

Q19. (C)

Sol.:



The required volume after rotation is

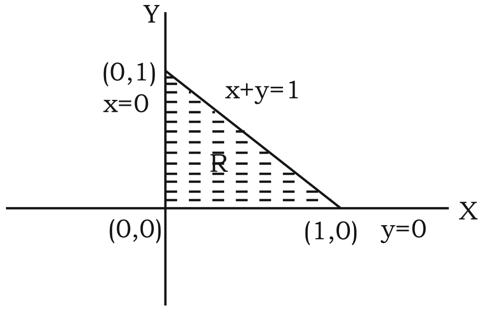
$$V = \pi\int_{-\sqrt{3}/2}^{\sqrt{3}/2} \pi(y^2 - 1/4)dz$$

$$= 2\pi\int_0^{\sqrt{3}/2}\left(\frac{3}{2}-z^2\right)dz.$$

$$= \frac{2\pi\sqrt{3}}{8}(3-1) = \frac{\pi\sqrt{3}}{2}$$

Q20. (B)

Sol.:



$\vec{F} = (3x-8y)\hat{i} + (4y-6xy)\hat{j}$

$\vec{r} = x\hat{i} + y\hat{j}$

$\Rightarrow \quad d\vec{r} = dx\hat{i} + dy\hat{j}$

$\vec{F}\cdot d\vec{r} = (3x-8y)dx + (4y-6xy)dy$

$\oint_C \vec{F}\cdot d\vec{r} = \oint_C (3x-8y)dx + (4y-6xy)dy$

Here, $P = 3x-8y$ & $Q = 4y-6xy$

$\Rightarrow \quad \frac{\partial P}{\partial y} = -8$ & $\frac{\partial Q}{\partial x} = -6y$

By Green's Theorem in plane

$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (-6y+8)dxdy$

$= \int_{x=0}^{1}\int_{y=0}^{1-x}(8-6y)dydx$

$$= \int_0^1 8y - \frac{6y^2}{2}\Big|_0^{1-x} dx$$

$$= \int_0^1 8(1-x) - 3(1-x)^2 dx$$

$$= -\frac{8(1-x)^2}{2}\Big|_0^1 + \frac{3(1-x)^3}{3}\Big|_0^1$$

$$= 4 - 1 = 3.$$

Q21. (C)

Sol.: $ST : U \to W$; $P : W \to U$

$T : U \to V$; $S : V \to W$

Rank (ST) = Nullity (P)

Nullity (ST) = Rank (P)

Rank (T) = Rank (S)

Now if dim(V) = 3 and dim U = 4

They by Rank - nullity theorem

dim (U) = Rank (ST) + Nullity (ST)

=Rank (P) + Nullity (P)

= dim (W) = 4

So, Rank (T) $\le 3 \Rightarrow$ Rank (ST) ≤ 3

$\Rightarrow$ Nullity (ST) $\ge 1 \Rightarrow$ Rank(P) ≥ 1

So P is not identically zero.

Q22. (A)

Sol.: $\frac{dy}{dx} + y = 0 \Rightarrow (D+1)y = 0 \Rightarrow y = ce^{-x}$

$(D+1)y = 2 \Rightarrow y = \left(\frac{1}{D+1}\right).2e^{0x} = 2$

$(D+1)y = 0 \Rightarrow y = \frac{1}{(D+1)}0 = 0$

$\Rightarrow$ $y(x) = ce^{-x} + 2$; $0 \le x < 1$

$= c_1 e^{-x}$; $x \ge 1$

$y(0) = 0 \Rightarrow c + 2 = 0 \Rightarrow c = -1$

For continuity of $y(x)$ we have

$ce^{-1} + 2 = c_1 e^{-1} \Rightarrow -\frac{2}{e} + 2 = \frac{c_1}{e}$

$\Rightarrow$ $c_1 = 2e - 2 = 2(e-1)$

$\Rightarrow$ $y(x) = 2(1 - e^{-x})$; $0 \le x < 1$

$= 2(e-1)e^{-x}$; $x \ge 1$

Q23. (C)

Sol.: For Differential equation

$$\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right)dx + \frac{1}{4}\left(x + xy^2\right)dy = 0$$

$M = y + \frac{1}{3}y^3 + \frac{1}{2}x^2$ &

$N = \frac{1}{4}(x + xy^2)$

$\Rightarrow$ $\frac{\partial M}{\partial y} = 1 + y^2$ & $\frac{\partial N}{\partial x} = \frac{1}{4}(1 + y^2)$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3}{x} = f(x)$$

So, Integrating factor is

$$e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3.$$

Q24. (B)

Sol.: $y'' + 3y' + 2y = e^{e^x}$

$\Rightarrow$ $(D^2 + 3D + 2)y = e^{e^x}$

$\Rightarrow$ $(D+2)(D+1)y = e^{e^x}$

Particular integral is

$$y = \frac{1}{(D+2)(D+1)}e^{e^x}$$

$$= e^{-2x}\int e^{(2x-x)}\int e^x.e^{e^x}dx$$

$$= e^{-2x}\int e^x.e^{e^x}dx$$

$$= e^{-2x}e^{e^x}$$

Q25. (A)

Sol.: Since number of group homomorphism

$f : Z_n \to Z_m$ is gcd(n,m)

Here, since there is only one homomorphism from

$G \to Z_{221}$ (Given)

Hence, $1 = gcd\left(o\left(G\right), 221\right)$

hence $G = Z_{21}$ among given options

Q26. (C)

Sol.: $H = \frac{Q}{Z}$ implies

I. Every cyclic subgroup of H is finite &
II. Every finite cyclic group is isomorphic to a subgroup of H.

Q27. (C)

Sol.: $x = \pm\sqrt{\frac{1 \pm \sqrt{5}}{2}} \Rightarrow x^2 = \frac{1 \pm \sqrt{5}}{2}$

$\Rightarrow \quad 2x^2 - 1 = \pm\sqrt{5} \Rightarrow (2x^2 - 1)^2 = 5$

$\Rightarrow \quad 4x^4 - 4x^2 - 4 = 0$

$\Rightarrow \quad x^4 = x^2 + 1 \qquad \ldots(i)$

$\Rightarrow \quad x^8 = (x^2 + 1)^2 = x^4 + 2x^2 + 1$

$= x^2 + 1 + 2x^2 + 1$

$= 3x^2 + 2$

Q28. (D)

Sol.: $\{(a, b) \in z^2 \mid 2$ divides a and 3 divides b$\}$

$= \{(2m, 3n) \mid m, n \in z\}$

is an abelian group w.r.t. '+'

Q29. (B)

Sol.: A, C, D are true but B is false because if

$f(x) = 1 \; ; \; x \in R \setminus Q$

$= -1 \; ; \; x \in Q$

Then if $J_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ then

$\lim_{n \to \infty} W(f, J_n) = 2$

Which is not equal to zero (0)

Q30. None are correct.

Q31. (B)

Sol.: $f(x) = x + \frac{1}{x^3} \Rightarrow f'(x) = 1 - \frac{3}{x^4}$

$\Rightarrow \quad f'(x) = \frac{x^4 - 3}{x^4}$

In $(0,1)$ $f'(x) < 0,$ so $f(x)$ is strictly decreasing, so f is one-one in $(0,1)$

But in other intervals $f'(x)$ is both positive as well as negative, so in those intervals it is not one-one.

Q32. (A, B, C)

Sol.: $\frac{dy}{dx} = (\sin 2x) y^{1/3}$ & $y(0) = 0$

$\Rightarrow \quad y(x) = 0$ is one of the solution

Now, $\frac{dy}{y^{1/3}} = \sin 2x \; dx$

$\Rightarrow \quad \int \frac{dy}{y^{1/3}} = \int \sin 2x \; dx$

$\Rightarrow \quad \frac{y^{2/3}}{2/3} = \sin^2 x + c$

$y(0) = 0 \Rightarrow c = 0$

$\Rightarrow \quad y^{2/3} = \frac{2}{3} \sin^2 x$

$\Rightarrow \quad y = \pm\sqrt{\frac{8}{27}} \sin^3 x.$

So, $y(x) = -\sqrt{\frac{8}{27}} \sin^3 x.$

So, $y(x) = -\sqrt{\frac{8}{27}} \sin^3 x.$

Q33. (A,D)

Sol.: $f \circ g \circ h \circ g \circ f(\alpha, \beta, \gamma, \delta)$

$= f \circ g \circ h \circ g(\beta, \alpha, \gamma, \delta)$

$= f \circ g \circ h(\gamma, \alpha, \beta, \delta)$

$= f \circ g(\delta, \alpha, \beta, \gamma)$

$= f(\delta, \alpha, \gamma, \beta) = (\delta, \beta, \gamma, \alpha)$

So, it interchanges δ & α but fixed β and γ.

also $h \circ g \circ f \circ g \circ h(\alpha, \beta, \gamma, \delta)$

$= h \circ g \circ f \circ g(\alpha, \beta, \delta, \gamma) = h \circ g \circ f(\alpha, \gamma, \delta, \beta)$

$= h \circ g(\beta, \gamma, \delta, \alpha) = h(\gamma, \beta, \delta, \alpha)$

$= (\delta, \beta, \gamma, \alpha)$

so it also interchanges δ & α but fixes β and γ.

Q34. (B,C,D)

Sol.: Union of two compact sets is again a compact set.

If P is set of rational and Q is set of irrationals, then both are not connected but $P \cup Q = R$ is connected.

$P \cup Q$ & P are closed need not emply that Q is closed as

$P = [0,1] \; ; \; Q = (0,1)$ & $P \cup Q = [0,1]$

$P \cup Q$ & P are open need not imply that Q is open as $P=(0,2), Q=[1,1.5]$ and $P\cup Q=(0,2)$.

Q35. (B,C,D)

Sol.: $e^{i\frac{2\pi}{93}}$ & $e^{i\frac{2\pi}{97}}\in\bigcup_{n=1}^{100}Y_n$ but their product

$e^{i\frac{4\pi}{93.97}}\notin\bigcup_{n=1}^{100}Y_n$

So, it is not a subgroup of c^* but rest all three are subgroups of c^*.

Q36. (A, B)

Sol.: The system is

$$\begin{bmatrix}1&1&1\\1&\beta&1\\1&1&\alpha\end{bmatrix}\begin{bmatrix}\alpha\\\gamma\\\beta\end{bmatrix}\sim\left[\begin{array}{ccc|c}1&1&1&\alpha\\0&\beta-1&0&\gamma-\alpha\\0&0&\alpha-1&\beta-\alpha\end{array}\right]$$

For the system to give solution, if $\beta-1=0$ then $\gamma-\alpha=0$

i.e., If $\beta=1$ then $\gamma=\alpha$.

Also, if $\alpha-1=0$ then $\beta-\alpha=0$

$\Rightarrow$ $\alpha=1$ then $\beta=1$ & $\beta=1$, then $\gamma=\alpha=1$.

Q37. (A,D)

Sol.: Rank(P) $\le$ m & Rank (Q) $\le$ m

So, Rank (PQ) $\le$ m & also Rank (QP) $\le$ m.
Hence Rank (PQ) cannot be n as $n>m$ and also Rank(QP) cannot be

$\left\lceil\frac{m+n}{2}\right\rceil$ as $\left\lceil\frac{n+m}{2}\right\rceil$

will be greater than m.

Q38. (A,B,C)

Sol.: $\vec{F}=(2x+3yz)\hat{i}+(3xz+2y)\hat{j}+(3xy+2z)\hat{k}$

$$\Rightarrow\nabla\times\vec{F}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\2x+3yz&3xz+2y&3xxy+2z\end{vmatrix}=0$$

$$\Rightarrow\quad\oint_C\vec{F}.d\vec{r}=\oiint_S(\nabla\times F).d\vec{s}=0$$

(By Stoke's theorem).

$\nabla\cdot\vec{F}=6=\phi_{xx}+\phi_{yy}+\phi_{zz}$; where

$\phi(x,y,z)=x^2+y^2+z^2$

Q39. (B,C,D)

Sol.: $x^2+x>4\Rightarrow x(x+1)>4$

As $x^2+x-4=0\Rightarrow x=\frac{-1\pm\sqrt{1+16}}{2}$

$=\frac{-1\pm\sqrt{17}}{2}$

So either $x<\frac{-1-\sqrt{17}}{2}$ or

$x>\frac{-1+\sqrt{17}}{2}$

$\frac{-1-\sqrt{17}}{2}$ $\frac{-1+\sqrt{17}}{2}$

So, $\{x\in R\mid x^2+x>4\}$ is not connected.

but $\{x\in R\mid x^2+x<4\}$ is connected

It is $\frac{-1-\sqrt{17}}{2}$ $\frac{-1+\sqrt{17}}{2}$

$\{x\in R\mid |x|<|x-4|\}\Rightarrow x<2$

2

So it is connected

& $\{x\in R\mid |x|>|x-4|\}\Rightarrow x>2$

so it is also connected.

Q40. (A,B,C)

Sol.: 2018 is interior point of $S\Rightarrow\exists\in>0$ s.t. $(2018-\in,2018+\in)\subseteq S$

and hence $2018+\frac{\in}{2}$ is also an interior point of S. Also there is a sequence in S which does not converge to 2018.

Q41. (4)

Sol.: (1 2 3) (2 4 5) (4 5 6)
= (1 5 2 3) (4 6)
So, order of (1 2 3) (2 4 5) (4 5 6)
= L.C.M. of 4 & 2 = 4

Q42. (7)

Sol.: $\phi(x,y,z)=3y^2+3yz$

$\Rightarrow$ grad$(\phi)=\nabla\phi=(6y+3z)j+3yk$

direction of given line is $2i-j-2k$. So directional derivative of $\phi(x,y,z)$ in the direction of given line is

$$\nabla\phi\cdot\hat{a}=\left[(6y+3z)j+3yk\right]\cdot\frac{\left[2i-j-2k\right]}{\sqrt{(2)^2+(-1)^2+(-2)^2}}$$

$$=\frac{-6y-3z-6y}{3}=-4y-z$$

$\Rightarrow$ directional derivative at $(1,-2,1)=7$

$\Rightarrow$ Absolute value of directional derivative = 7

Q43. (1)

Sol.: $f(x)=\sum_{n=0}^{\infty}(-1)^n x(x-1)^n$

$$=x\left[1-(x-1)+(x-1)^2-\ldots\right]$$

$$=\frac{x}{1--(x-1)}=\frac{x}{x}=1.$$

$\Rightarrow$ $f\left(\frac{\pi}{4}\right)=1.$

Q44. (1)

Sol.:

$$f(x,y)=\begin{cases}\dfrac{x^2y(x-y)}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$\Rightarrow f_x(0,0)=\underset{h\to0}{lt}\dfrac{f(h,0)-f(0,0)}{h}=0$

$$=f_y(0,0)=\underset{k\to0}{lt}\frac{f(0,k)-f(0,0)}{k}=0$$

$$f_{xy}(0,0)=\underset{h\to0}{lt}\frac{f_y(h,0)-f_y(0,0)}{h}$$

$$=\underset{h\to0}{lt}\frac{\underset{k\to0}{lt}\dfrac{f(h,k)-f(h,0)}{k}}{h}$$

$$=\underset{h\to0}{lt}\frac{\underset{k\to0}{lt}\left(\dfrac{h^2k(h-k)}{h^2+k^2}\right)/k}{h}=1$$

$$=f_{yx}(0,0)=\underset{k\to0}{lt}\frac{f_x(0,k)-f_x(0,0)}{k}$$

$$=\underset{k\to0}{lt}\frac{\underset{k\to0}{lt}\dfrac{f(h,k)-f(0,k)}{h}}{k}$$

$$\underset{k\to0}{lt}\frac{h^2k(h-k)}{\dfrac{h^2+k^2}{h}}$$

$$=\underset{k\to0}{lt}\frac{h}{k}=0$$

$\Rightarrow$ $f_{xy}(0,0)-f_{yx}(0,0)=1.$

Q45. (3)

Sol.: $f(x,y)$ is homogeneous function in x & y of degree 2, so by Euler's theorem.

$f_x(x,y)+f_y(x,y)=2f(x,y)$

$\Rightarrow$ $f_x(1,1)+f_y(1,1)=2f(1,1)$

$$=2\left[1+\frac{1}{2}\right]=2\left[\frac{3}{2}\right]=3.$$

Q46. (9)

Sol.: $f(x)=\int_0^x\sqrt{f(t)}dt$

$\Rightarrow$ $f'(x)=\sqrt{f(x)}$ & $f(0)=0$

$\Rightarrow$ $\dfrac{df(x)}{\sqrt{f(x)}}=dx$

$\Rightarrow$ $2\sqrt{f(x)}=x+c\Rightarrow 2\sqrt{f(x)}=x$

$\Rightarrow$ $f(x)=\left(\dfrac{x}{2}\right)^2\Rightarrow f(6)=\left(\dfrac{6}{2}\right)^2=9$

Q47. (2)

Sol.: limit points of $\left[\dfrac{(1+(-1)^n)}{2^n}\right]^{1/n}$ are

$0\ \&\ \frac{1}{2}\cdot$ so limit superior will be $1/2$

limit points of $\left[\dfrac{(1+(-1)^{n-1})}{3^n}\right]^{1/n}$ are 0 & $\frac{1}{3}$ so

limit suprerior will be $1/3$.

Radius of convergence of power series

$\sum\dfrac{(1+(-1)^n)}{2^n}x^n$ is $\dfrac{1}{1/2}=2$

Radius of convergence of power series

$\sum\dfrac{(1+(-1)^{n-1})}{3^n}x^n$ is $\dfrac{1}{1/3}=3$

So, Radius of convergence of

$\sum a_n x^n$ will be $\min(2,3)=2$

Q48. (0)

Sol.: There will be no element of order 6

Q49. (4)

Sol.: $W_1 \cap W_2$ will be vector space of all 5×2 matrices whose each row and each column sums to 0.

So dimension of $W_1 \cap W_2$ is

$(5-1)(2-1)=(4)(1)=4.$

Q50. (0.125)

Sol.: coefficient of x^4 in the power series expansion of $e^{\sin x}=f(x)$ about x = 0 is

$$\frac{f^{iv}(0)}{4!}$$

Now $f(x)=e^{\sin x}$

$\Rightarrow \quad f'(x)=\cos x e^{\sin x}$

$\Rightarrow \quad f''(x)=e^{\sin x}(\cos^2 x-\sin x)$

$\Rightarrow \quad f'''(x)=e^{\sin x}(\cos^3 x-3\cos x\sin x-\cos x)$

$$\Rightarrow \quad f^{iv}(x)=e^{\sin x}\begin{bmatrix}\cos^4 x-3\cos^2 x\sin x-\cos^2 x\\ -3\cos^2 x\sin x+\sin x\\ +3\sin^2 x-3\cos^2 x\end{bmatrix}$$

$\Rightarrow \quad f^{iv}(0)=1-1-3=-3$

$$\Rightarrow \quad \frac{f^{iv}(0)}{4!}=\frac{-3}{24}=-\frac{1}{8}=-0.125$$

Q51. (0.5)

Sol.: $a_k=(-1)^{k-1}\Rightarrow S_n=a_1+a_2+\dots+a_n$

$\Rightarrow \quad S_n=1-1+1-1+\dots$

$\Rightarrow \quad S_n=1$ is n is odd

$\quad = 0$ if n is even

$$\Rightarrow \quad \sigma_n=\frac{(S_1+S_2+\dots+S_n)}{n}$$

$$=\frac{\frac{n}{2}}{n}=\frac{1}{2},\ \text{if n is even}$$

$$=\frac{n+1}{2n}=\frac{1}{2}+\frac{1}{2n},\ \text{if n is odd}$$

So, $\lim_{n\to\infty}\sigma_n=\frac{1}{2}=0.5$

Q52. (0.368)

Sol.: $$\lim_{x\to\infty}\left(f\left(\sqrt{\frac{2}{x}}\right)\right)^x \left[1^\infty \text{ case}\right]$$

$$=e^{\lim_{x\to\infty} x\left(f\left(\sqrt{\frac{2}{x}}\right)-1\right)}$$

$$=e^{\lim_{x\to\infty}\frac{f\left(\sqrt{\frac{2}{x}}\right)-1}{(1/x)}}\left[\frac{0}{0}\text{case}\right]$$

$$=e^{\lim_{x\to\infty}\frac{\left[f'\left(\sqrt{\frac{2}{x}}\right)\right]\frac{1}{2}\times 2}{\sqrt{\frac{2}{x}}}};\left[\frac{0}{0}\text{case}\right]$$

$$=e^{\lim_{x\to\infty} f''\left(\sqrt{\frac{2}{x}}\right)}=e^{f''(0)}=e^{-1}$$

$$=\frac{1}{e}=0.367879\approx 0.368$$

Q53. (1152)

Sol.: By Lagrangian multiplier method

Let $F(x,y,z)=xyz^2+\lambda(x+2y+3z-1)$

$\Rightarrow \quad F_x=yz^2+\lambda=0 \Rightarrow yz^2=-\lambda \quad \dots(i)$

$F_y=xz^2+2\lambda=0\Rightarrow xz^2=-2\lambda \quad \dots(ii)$

$F_z=2xyz+3\lambda=0\Rightarrow 2xyz=-3\lambda \quad \dots(iii)$

$F_\lambda=x+2y+3z-1=0\Rightarrow x+2y+3z=1 \dots(iv)$

(i) & (ii) $\Rightarrow \frac{y}{x}=\frac{1}{2}\Rightarrow x=2y$

(ii) & (iii) $\Rightarrow \frac{z}{2y}=\frac{2}{3}\Rightarrow z=\frac{4y}{3}$

putting in (4), we get

$2y+2y+4y=1\Rightarrow 8y=1\Rightarrow y=\frac{1}{8}$

$\Rightarrow \quad x=\frac{1}{4}; z=\frac{1}{6}$

So, maximum value of $xyz^2=M$ is

$$M=\left(\frac{1}{4}\right)\left(\frac{1}{8}\right)\left(\frac{1}{6}\right)^2=\frac{1}{(32)(36)}$$

$$= \frac{1}{1152}$$

$$\Rightarrow \quad \frac{1}{M} = 1152$$

Q54. (6)

Sol.: Volume,

$$V = \int_{-1}^{1}\int_{-1}^{1}\int_{\sqrt{2-y^2}}^{2} dzdydx$$

$$= \int_{-1}^{1}\int_{-1}^{1} 2 - \sqrt{2-y^2}\,dydx$$

$$= 2\int_{-1}^{1} 2y - \left[\frac{y}{2}\sqrt{2-y^2} + \frac{2}{2}\sin^{-1}\frac{y}{\sqrt{2}}\right]\Bigg|_0^1 dx$$

$$= 2\int_{-1}^{1} 2 - \left[\frac{1}{2} + \frac{\pi}{4}\right]dx$$

$$= 2\left(\frac{3}{2} - \frac{\pi}{4}\right)\int_{-1}^{1} dx$$

$$= 4\left(\frac{3}{2} - \frac{\pi}{4}\right) = 6 - \pi$$

Now $V = \alpha - \pi = 6 - \pi$

$$\Rightarrow \quad \alpha = 6.$$

Q55. (3)

Sol.:

method 1 : $\alpha = \int_{\pi/6}^{\pi/3} \frac{\sin t + \cos t}{\sqrt{\sin 2t}}\,dt$,

put y = sint -cost $\Rightarrow$ dy = (sint + cost) dt

so, $\sqrt{\sin 2t} = \sqrt{1 - x^2}$

$$\Rightarrow \quad \alpha = \int_{(1-\sqrt{3})/2}^{(\sqrt{3}-1)/2} \frac{1}{\sqrt{1-y^2}}\,dy = 2\sin^{-1}\left((\sqrt{3}-1)/2\right)$$

$$\Rightarrow \sin(\alpha/2) = \frac{\sqrt{3}-1}{2} \quad \Rightarrow \left(2\sin\frac{\alpha}{2} + 1\right)^2 = 3$$

method 2 : $\alpha = \int_{\pi/6}^{\pi/3} \frac{\sin t + \cos t}{\sqrt{\sin 2t}}\,dt$

$$= \frac{1}{\sqrt{2}}\int_{\pi/6}^{\pi/3} \sqrt{\tan t} + \sqrt{\cos t}\ dt$$

$$= \sqrt{2}\int_{\pi/6}^{\pi/3} \sqrt{\tan t}\ dt$$

Now, take, $I = \int \sqrt{\tan t}\ dt$

Let $x^2 = \tan t \Rightarrow 2xdt = \sec^2 tdt$

$$\Rightarrow \quad dt = \frac{2x}{1+x^4}\,dx$$

$$\Rightarrow \quad I = \int \frac{2x^2}{1+x^4}\,dx = \int \frac{(x^2+1)+(x^2-1)}{1+x^4}\,dx$$

$$= \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx + \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$= \int \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2} + \int \frac{d(x+\frac{1}{x})}{\left(x+\frac{1}{x}\right)^2-2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\ln\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{\tan t}-\sqrt{\cos t}}{\sqrt{2}}\right)$$

$$+\frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{\tan t}+\sqrt{\cos t}-\sqrt{2}}{\sqrt{\tan t}+\sqrt{\cos t}+\sqrt{2}}\right)$$

$$\Rightarrow \quad \alpha = \left[\begin{array}{l}\tan^{-1}\left(\frac{\sqrt{\tan t}-\sqrt{\cos t}}{\sqrt{2}}\right)+ \\ \frac{1}{2}\ln\left(\frac{\sqrt{\tan t}+\sqrt{\cos t}-\sqrt{2}}{\sqrt{\tan t}+\sqrt{\cos t}+\sqrt{2}}\right)\end{array}\right]_{\pi/6}^{\pi/3}$$

$$\Rightarrow \quad \alpha = 2\tan^{-1}\frac{\sqrt{3}-1}{\sqrt{2\sqrt{3}}}$$

$$\Rightarrow \quad \frac{\alpha}{2} = \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

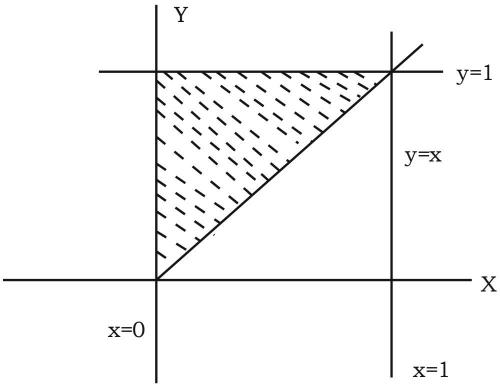
$$\Rightarrow \quad \sin\frac{\alpha}{2} = \frac{\sqrt{3}-1}{2}$$

$$\Rightarrow \quad 2\sin\frac{\alpha}{2} + 1 = \sqrt{3}$$

$$\Rightarrow \quad \left(2\sin\frac{\alpha}{2}+1\right)^2 = 3$$

Q56. (0.239)

Sol.:



Let $I = \int_0^1 \int_x^1 y^4 e^{xy^2} dy dx$

By changing order of Integration

$$I = \int_{y=0}^1 \int_{x=0}^y y^4 e^{xy^2} dx dy =$$

$$= \int_{y=0}^1 \frac{y^4 e^{xy^2}}{y^2} \Big|_0^y dy = \int_0^1 y^2 \left(e^{y^3} - 1\right) dy$$

$$= \frac{e^{y^3}}{3} - \frac{y^3}{3} \Big|_0^1$$

$$= \frac{1}{3}(e-1) - \frac{1}{3} = \frac{1}{3}(e-2)$$

$$= \frac{1}{3}(2.71828 - 2) = \frac{0.71828}{3}$$

$$= 0.2394267 \approx 0.239$$

Q57. (6)

Sol.: Rank(Q) = 2

Now T(P) = QP

So, $\{P \in M_{3\times3}(R) \text{ s.t. } T(P) = QP = 0\}$

$= N(T)$

$\Rightarrow$ Nullity (T) = 9-3 ; Rank (Q) = 9-3(2) = 9-6 = 3

$\Rightarrow$ Rank(T) = 9-3 = 6

By Rank Nullity Theorem.

Q58. (6.51)

Sol.: $\phi(u,v) = (2+\cos u)\cos v\hat{i} + (2+\cos u)\sin v\hat{j} + \sin u\hat{k}$

$$\Rightarrow \frac{\partial\phi}{\partial u} \times \frac{\partial\phi}{\partial v}$$

$$= \begin{vmatrix} i & j & k \\ -\sin u\cos v & -\sin u\sin v & \cos u \\ -(2+\cos u)\sin v & (2+\cos u)\cos v & 0 \end{vmatrix}$$

$$= -(2+\cos u)\cos u\cos v i + (2+\cos u)\sin v\cos u$$

$$+(-(2+\cos u)\sin u)$$

$$\Rightarrow \left\|\frac{\partial\phi}{\partial u} \times \frac{\partial\phi}{\partial v}\right\| = 2+\cos u$$

So, total area =

$$\int_0^{\pi/2}\int_0^{\pi/2} (2\cos u) du dv$$

$$= \int_0^{\pi/2} \left[2(\pi/2) + 1\right] dv$$

$$= (\pi+1)\frac{\pi}{2} \approx \frac{29}{7} \times \frac{22}{14} = \frac{319}{49}$$

$= 6.51$ (approx).

Q59. (-2.72)

Sol.: $\frac{dx}{dt} = x^2t^3 + xt$

$$\Rightarrow \frac{dx}{dt} - tx = t^3x^2$$

$$\Rightarrow x^{-2}\frac{dx}{dt} = tx^{-1} = t^3$$

Let $x^{-1} = y \Rightarrow -x^{-2}\frac{dx}{dx} = \frac{dy}{dt}$

$$\Rightarrow -\frac{dy}{dt} - ty = t^3$$

$$\Rightarrow \frac{dy}{dt} + ty = -t^3$$

$$\Rightarrow ye^{t^2/2} = \int t^3 e^{t^2/2} dt + c$$

$$\Rightarrow ye^{t^2/2} = -\left(2\cdot\frac{t^2}{2} - 2\right)e^{t^2/2} + c$$

$$\Rightarrow \frac{1}{x} = -(t^2-2) + ce^{-t^2/2}$$

$$\Rightarrow x(0) = 1 \Rightarrow 1 = +2 + c \Rightarrow c = -1$$

$$\Rightarrow \frac{1}{x} = -(t^2-2) - e^{-t^2/2}$$

$$\Rightarrow \frac{1}{x\left(\sqrt{2}\right)} = -e^{-1} \Rightarrow x\left(\sqrt{2}\right) = -e$$

$$= -2.71828 \approx -2.72$$