Introduction to Ordinary Differential Equations and Some Applications

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Introduction

These notes are my attempt to present the necessary material for Math 46 at UC Riverside, as well as some further topics that I believe are interesting. I have done my best to make things as clear as possible, however I am sure that there are imperfections in these notes, be they typos, grammar errors, etc. That being so, I warmly welcome any suggestions that you may have to make these better.

My main motivation for writing these notes stems from the outrageous price for a book that is only used for one quarter. As of Spring 2010 the used price for the "official" book is $150 and the new price is $200, and I believe that students should not have to pay such a big price for a book that is used so briefly. Also, in a class such as Math 46, I believe that the textbook should be readable and not too technical, which is exactly what the current text is not.

Hopefully these notes are as useful as I intend them to be.

0. Prerequisites

This chapter is meant to give an overview of the material you should have learned at some point in the prerequisite courses. However, there is material in this chapter that you might not have encountered yet. The material that is in the prerequisite courses is mostly just included as a reminder, but the material that might be new has more than just examples.

0.1. Precalculus Topics.

0.1.1. *Partial Fractions.*

Partial fractions is a topic that seems to haunt students because they perceive it as hard because it involves more than just trivial plug and chug. However it is quite an algorithmic process, and as such, it is really not as difficult as most people might think.

By the **Fundamental Theorem of Algebra** we can factor any polynomial into a product of linear terms and irreducible quadratics. We will use this fact to decompse rational functions by the method of **Partial Fractions**. The process of partial fractions is essentially the reverse process of adding fractions. I believe that the best way to accomplish the understanding of the process of partial fractions is through examples.

Linear Terms in the Denominator

These are examples that contain no irreducible quadratics in the denominator.

Distinct Terms

Example 1. *Decompose* $\dfrac{x-7}{x^2+2x-15}$ *into partial fractions.*

Solution. *First we should factor the bottom:*

$$x^2+2x-15=(x+5)(x-3).$$

Now rewrite the fraction with the factored bottom:

$$\frac{x-7}{x^2+2x-15}=\frac{x-7}{(x+5)(x-3)}.$$

Next we perform the operation of partial fractions:

$$\frac{x-7}{(x+5)(x-3)}=\frac{A}{x+5}+\frac{B}{x-3}.$$

Then we solve for A and B. To do this we multiply through by the denominator on the left side of the equality above (this is actually the LCD of the fractions on the right side). Multiplying through and gathering like terms we get:

$$x-7=A(x-3)+B(x+5)=(A+B)x+(-3A+5B).$$

Now equating coefficients of like terms we get the system of equations:

$$\begin{cases} A + B = 1 \\ -3A + 5B = -7 \end{cases}$$

Which can easily be solved to get:

$$A=\frac{3}{2} \text{ and } B=-\frac{1}{2}$$

Thus the partial fraction decomposition is:

$$\frac{x-7}{x^2+2x-15}=\frac{\frac{3}{2}}{x+5}-\frac{\frac{1}{2}}{x-3}.$$

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Repeating Terms

Example 2. *Decompose* $\dfrac{2x^2+5}{(x+2)^2(x-1)}$ *into partial fractions.*

Solution. *This looks a bit more complex than the last example, but have no fear, we can still do it! First, as before, rewrite the fraction in its decomposed form:*

$$\frac{2x^2+5}{(x+2)^2(x-1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-1}.$$

Multiplying through by the denominator on the left and gathering like terms we get:

$$\begin{aligned} 2x^2+5 &= A(x+2)(x-1) + B(x-1) + C(x+2)^2 \\ &= A(x^2+x-2) + B(x-1) + C(x^2+4x+4) \\ &= (A+C)x^2 + (A+B+4C)x + (-2A-B+4C) \end{aligned}$$

This gives us a system of equations which, as before, except this is a 3×3 system:

$$\begin{cases} A \quad\quad\; + C = 2 \\ A + B + 4C = 0 \\ -2A - B + 4C = 5 \end{cases}$$

Adding together all three equations we get:

$$9C = 7 \quad \Rightarrow \quad C = \frac{7}{9}$$

Using the first equation we get:

$$A = 2 - C = 2 - \frac{7}{9} = \frac{11}{9}$$

And the second equations gives:

$$B = -A - 4C = -\frac{11}{9} - \frac{28}{9} = -\frac{39}{9} = -\frac{13}{3}$$

Thus the decomposition is:

$$\frac{2x^2+5}{(x+2)^2(x-1)} = \frac{\frac{11}{9}}{x+2} - \frac{\frac{13}{3}}{(x+2)^2} + \frac{\frac{7}{9}}{x-1}.$$

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Quadratic Terms in the Denominator

There are many different combinations involving quadratic terms we can have in the denominator at this point: distinct linear and quadratic terms, repeating linear terms and distinct quadratic terms, no linear terms and distinct quadratic terms, no linear terms and repeating quadratic terms, distinct linear terms and repeating quadratic terms, and repeating linear and quadratic terms. Here I will give an example of the first and fourth one:

Distinct Linear and Quadratic Terms

Example 3. *Decompose* $\frac{5x^2-7x+15}{(2x+1)(4x^2+3)}$ *into partial fractions.*

Solution. *As always, begin by decomposing the fraction:*

$$\frac{5x^2-7x+15}{(2x+1)(4x^2+3)} = \frac{A}{2x+1} + \frac{Bx+C}{4x^2+3}$$

Then multiply through by the LCD and gather like terms:

$$\begin{aligned} 5x^2-7x+15 &= A(4x^2+3) + (Bx+C)(2x+1) \\ &= (4A+2B)x^2 + (B+2C)x + (3A+C) \end{aligned}$$

Equating coefficients we get the system:

$$\begin{cases} 4A + 2B = 5 \\ B + 2C = -7 \\ 3A + C = 15 \end{cases}$$

You can check that we have:

$$A = \frac{79}{16} \quad B = -\frac{59}{8} \quad C = \frac{3}{16}$$

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No Linear Terms and Repeating Quadratic Terms

Example 4. *Decompose* $\dfrac{8s^3+13s}{(s^2+2)^2}$ *into partial fractions.*

Solution. *Decompose:*

$$\frac{8s^3+13s}{(s^2+2)^2} = \frac{As+B}{s^2+2} + \frac{Cs+D}{(s^2+2)^2}$$

Multiply by the LCD and collect like terms:

$$\begin{aligned} 8s^3+13s &= (As+B)(s^2+2)+(Cs+D) \\ &= As^3+2As+Bs^2+2B+Cs+D \\ &= (A)s^3+(B)s^2+(2A+C)s+(D) \end{aligned}$$

Thus we have the system of equations:

$$\begin{cases} A & & & & = & 8 \\ & B & & & = & 0 \\ 2A & & + \; C & & = & 13 \\ & & & D & = & 0 \end{cases}$$

It is incredibly easy to see that:

$$A=8 \quad B=0 \quad C=-3 \quad D=0$$

And so the decomposition is:

$$\frac{8s^3+13s}{(s^2+2)^2} = \frac{8s}{s^2+2} - \frac{3s}{(s^2+2)^2}$$

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0.1.2. *Trigonometric Identities.*

Trigonometry... It seems to be the main shortcoming of many students... Some people don't do well with it because they think that they need to memorize all these different formulas, however you really only need to know one, the sum formula for cos:

$$\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v).$$

Here is an extensive, but not complete, list of the formulas, mostly intended for quick reference:

a. $\sin^2\theta + \cos^2\theta = 1$
b. $1+\tan^2\theta = \sec^2\theta$
c. $1+\cot^2\theta = \csc^2\theta$
d. $\sin(-\theta) = -\sin(\theta)$ (i.e. sin is odd)
e. $\cos(-\theta) = \cos(\theta)$ (i.e. cos is even)
f. $\sin(\theta - \frac{\pi}{2}) = -\cos(\theta)$
g. $\cos(\theta - \frac{\pi}{2}) = \sin(\theta)$
h. $\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v)$
i. $\cos(u-v) = \cos(u)\cos(v) + \sin(u)\sin(v)$
j. $\sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)$
k. $\sin(u-v) = \sin(u)\cos(v) - \cos(u)\sin(v)$
l. $\sin(2\theta) = 2\sin\theta\cos\theta$
m. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
n. $\cos(2\theta) = 2\cos^2\theta - 1$
o. $\cos(2\theta) = 1 - 2\sin^2\theta$
p. $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$
q. $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$

To prove my point I will prove some of the above identities using identity h:

h ⇒ a)

$$\begin{aligned} 1 = \cos(0) &= \cos(x-x) \\ &= \cos[x+(-x)] \\ &= \cos(x)\cos(-x) - \sin(x)\sin(-x) \\ &= \cos(x)\cos(x) - \sin(x)(-\sin(x)) \\ &= \cos^2 x + \sin^2 x \end{aligned}$$

h $\Rightarrow$ f) First note that $\sin\left(\theta - \frac{\pi}{2}\right) = -\cos(\theta)$ is equivalent to $-\sin(\theta - \frac{\pi}{2}) = \cos(\theta)$. Then:

$$\begin{aligned}
\cos(\theta) &= \cos\left[\left(\theta - \frac{\pi}{2}\right) + \frac{\pi}{2}\right] \\
&= \cos\left(\theta - \frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\theta - \frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) \\
&= \cos\left(\theta - \frac{\pi}{2}\right)(0) - \sin\left(\theta - \frac{\pi}{2}\right)(1) \\
&= -\sin\left(\theta - \frac{\pi}{2}\right)
\end{aligned}$$

Exercises.

Find the partial fraction decomposition of the following fractions:

(1) $\dfrac{-x+5}{(x-1)(x+1)}$

(2) $\dfrac{2s+1}{s^2+s}$

(3) $\dfrac{2v-3}{v^3+10v}$

(4) $\dfrac{-4x}{3x^2-4y+1}$

(5) $\dfrac{2s+2}{s^2-1}$

(6) $\dfrac{4z^2+3}{(z-5)^3}$

(7) $\dfrac{2x^2-6}{x^3+x^2+x+1}$

Prove the following identities:

(8) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

(9) $\tan(u-v) = \dfrac{\tan u - \tan v}{1 - \tan u \tan v}$

(10) $\sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$

(11) $\cos u \sin v = \frac{1}{2}[\sin(u+v) - \sin(u-v)]$

(12) $\dfrac{\cos t}{1-\sin t} = \dfrac{1+\sin t}{\cos t}$

(13) $\sec\theta - \cos\theta = \tan\theta\sin\theta$

(14) $\cos^4 x - \sin^4 x = \cos 2x$

0.2. **Differentiation Techniques.**

0.2.1. *The Leibniz Rule.*

The Leibniz rule (a.k.a. the product rule) is arguably one of the most important rules involving differentiation. While I cannot do justice to it here, I will present it with an example of its application.

Theorem 1 (The Leibniz Rule)**.** *Suppose that f and g are differentiable functions, then:*

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

Example 1. *Differentiate:*

$$f(x) = e^x \sin(x)$$

Solution. *Simply follow the Leibniz rule:*

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[(e^x)(\sin(x))\right] \\ &= \frac{d}{dx}\left[e^x\right]\sin(x) + e^x\frac{d}{dx}\left[\sin(x)\right] \\ &= e^x\sin(x) + e^x\cos(x) \end{aligned}$$

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0.2.2. *Chain Rule.*

The chain rule is another one of the most important rules regarding differentiation. This rule deals with taking derivatives of compositions of functions.

Theorem 2. *Suppose that f and g are differentiable functions and the composition $f \circ g$ makes sense. Then:*

$$(f \circ g)'(x) = (f(g(x)))' = f'(g(x)) \cdot g'(x).$$

In other notation, if $y = f(u)$ and $u = g(x)$ with f and g differentiable, then the chain rule can be written as:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

The second way of writing this is useful for implicit differentiation.

Example 2. *Find the derivative of:*

$$h(x) = \sqrt{x^2+1}.$$

Solution.

$$\begin{aligned} h'(x) &= \frac{d}{dx}\sqrt{x^2+1} \\ &= \frac{1}{2}\frac{1}{\sqrt{x^2+1}} \cdot \frac{d}{dx}\left(x^2+1\right) \\ &= \frac{2x}{2\sqrt{x^2+1}} \\ &= \frac{x}{\sqrt{x^2+1}} \end{aligned}$$

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0.2.3. *Implicit Differentiation.*

Sometimes we would like to take a derivative of an equation that is not a function, such as $x^2 + y^2 = 1$, or even more fancy $y^5 + x^2y^3 = 1 + x^4y$. Thankfully, due to the chain rule, if we think of y as a function of x, we can still differentiate these equations using a method known as **Implicit Differentiation**. Let's take the derivative of the second equation above:

Example 3. *Find $\frac{dy}{dx}$ if:*

$$y^5 + x^2y^3 = 1 + x^4y.$$

Solution. *By implicit differentiation and the Leibniz rule we have:*

$$5y^4\frac{dy}{dx} + 2xy^3 + x^2(3y^2)\frac{dy}{dx} = 4x^3y + x^4\frac{dy}{dx}.$$

Now gather all terms with $\frac{dy}{dx}$ *in them on one side, and everything else on the other side:*

$$5y^4\frac{dy}{dx} + 3x^2y^2\frac{dy}{dx} - x^4\frac{dy}{dx} = 4x^3y - 2xy^3.$$

Factor out $\frac{dy}{dx}$ *and isolate it:*

$$\begin{aligned}\frac{dy}{dx}\left(5y^4 + 3x^2y^2 - x^4\right) &= 4x^3y - 2xy^3 \\ \frac{dy}{dx} &= \frac{4x^3y - 2xy^3}{5y^4 + 3x^2y^2 - x^4}\end{aligned}$$

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Exercises. Find the derivative of the following functions:

(1) $f(x) = \left(x^3 + 2x^2 - 3\right)^4$

(2) $y = \sin(\cos(x))$

(3) $g(x) = \frac{1}{\sqrt{x^2+1}}$

(4) $h(x) = \frac{(f(x))^3}{\sqrt{g(x)}}$, where f and g are differentiable functions

Use implicit differentiation to find y':

(5) $x^2 + y^2 = 16$

(6) $x^3\cos(y) + y^3\sin(x) = 9$

(7) $\sin(xy) = x^2 - y$

Further study:

(8) Notice that I made no mention of the so called *quotient rule* above. That is because it is a consequence of the Leibniz and chain rules and not really a "rule" in this respect. In this exercise you will prove the quotient rule using the Leibniz and chain rules.
Suppose that f and g are differentiable functions and that $g(x) \neq 0$ for any real number x. Show that

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

by rewriting $\frac{f(x)}{g(x)}$ as $f(x)\cdot[g(x)]^{-1}$, then taking the derivative.

(9) Prove the "Triple Product Rule": Let f, g, and h be differentiable functions, then

$$\frac{d}{dx}\left[f(x)g(x)h(x)\right] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

(10) Verify the following equation for differentiable functions f and g:

$$\frac{d^2}{dx^2}\left[f(x)g(x)\right] = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x).$$

0.3. Integration Techniques.

0.3.1. *u-Substitution.*

The (usual) first and easiest of the typical integration techniques is known as **u-Substitution**. u is really just a generic "dummy" variable, in fact this technique is really just a consequence of something known as "change of variables".

Theorem 1 (u-Substitution)**.** *Suppose $u = g(x)$ is a differentiable function whose range is an interval, and that f is a function that is continuous on the range of g, then*

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Example 1. *Find the following integral:*

$$\int \tan(x)\,dx.$$

Solution. *Since*

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx,$$

if we let $u = \cos(x)$, then $du = -\sin(x)$, and so we have:

$$\begin{aligned}\int \frac{\sin(x)}{\cos(x)}\,dx &= -\int \frac{1}{u}\,du \\ &= -\ln|u| + C.\end{aligned}$$

So substituting back in for u we get:

$$\int \tan(x)\,dx = -\ln|\cos(x)| + C.$$

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0.3.2. *Integration by Parts.*

For derivatives we have the the product rule. Recall that integration is an attempt to go backwards from differentiation. So you might hope we have a backward process for the product rule... This is more or less true, and this technique is called *Integration by Parts.*

Theorem 2 (Integration by Parts)**.** *Suppose that f and g are differentiable functions and that f' and g' are integrable functions. Then*

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

A more typical presentation of the integration by parts technique is the following: with f and g as in the theorem above, let $u = f(x)$ and $v = g(x)$, then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, so by substituting into the above equation we get:

$$\int u\,dv = uv - \int v\,du.$$

Example 2. *Find the following integral:*

$$\int x\cos(x)\,dx.$$

Solution. *Let $u = x$ and $dv = \cos(x)\,dx$ (a word of wisdom here, do not forget to include the dx when you choose dv, otherwise you won't be able to find v because you can't integrate!), then $du = dx$ and $v = \sin(x)$. So using integration by parts we get:*

$$\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx = x\sin(x) + \cos(x) + C.$$

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0.3.3. *Trigonometric Substitution.*

The technique of *Trigonometric Substitution* (or "trig sub" for short) is mainly based off of the following two trig identities:

$$\sin^2\theta + \cos^2\theta = 1,$$

and

$$1 + \tan^2\theta = \sec^2\theta.$$

Suppose we had an integral involving one of the three expressions: $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, or $\sqrt{x^2-a^2}$, where a is any nonzero positive real number. How would we solve these integrals? u-substitution won't work, integration by parts is hopeless... so what do we do? Notice that the above two identities can be rewritten as:

$$(a\sin\theta)^2 + (a\cos\theta)^2 = a^2,$$

and

$$a^2 + (a\tan\theta)^2 = (a\sec\theta)^2.$$

We can actually use this to our advantage, for example if we let $x = a\sin\theta$ the integral $\int\sqrt{a^2-x^2}\,dx$ becomes $\int a\cos\theta\sqrt{a^2-(a\sin\theta)^2}\,dx\theta = \int a\cos\theta\sqrt{(a\cos\theta)^2}\,d\theta = \int a^2\cos^2\theta\,d\theta$ which is a managable integral. There is a minor technicality we need to worry about here, and that is that the substitution is one-to-one, which can be achieved by demanding that θ lie in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. The following table summarizes the trig substitutions ($a>0$):

Expression in Integral	Substitution to Use
$\sqrt{a^2-x^2}$	$x = a\sin\theta,\quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$
$\sqrt{a^2+x^2}$	$x = a\tan\theta,\quad -\frac{\pi}{2}<\theta<\frac{\pi}{2}$
$\sqrt{x^2-a^2}$	$x = a\sec\theta,\quad 0\le\theta<\frac{\pi}{2}$ or $\pi\le\theta<\frac{3\pi}{2}$

Example 3. *Find the integral:*

$$\int x^3\sqrt{9-x^2}\,dx.$$

Solution. *Seeing the expression* $\sqrt{9-x^2}$ *suggests that we should make a substitution of the form* $x = 3\sin\theta$ *where* $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. *Plugging this in the integral we get:*

$$\begin{aligned}\int x^3\sqrt{9-x^2}\,dx &= \int(3\sin\theta)^3\sqrt{9-(3\sin\theta)^2}\,dx\\ &= \int 27\sin^3\theta\sqrt{9\cos^2\theta}(3\cos\theta)\,d\theta\\ &= 81\int\sin^3\theta\cos^2\theta\,d\theta\\ &= 81\int\sin\theta\sin^2\theta\cos^2\theta\,d\theta\\ &= 81\int\sin\theta(1-\cos^2\theta)\cos^2\theta\,d\theta\\ &= 81\int\sin\theta(\cos^2\theta-\cos^4\theta)\,d\theta\end{aligned}$$

Now make the substitution $u=\cos\theta$ *to get:*

$$81\int\sin\theta(\cos^2\theta-\cos^4\theta)\,d\theta = -81\int u^2-u^4\,du = -81\left(\frac{u^3}{3}-\frac{u^5}{5}\right)+C = -27u^3+\frac{81}{5}u^5+C = -27\cos^3\theta+\frac{81}{5}\cos^5\theta+C.$$

Now $\theta = \arcsin\frac{x}{3}$ *and* $\cos\left(\arcsin\frac{x}{3}\right) = \frac{1}{3}\sqrt{9-x^2}$ *(to see this last equality, draw a triangle) so we (finally) get:*

$$-27\cos^3\theta+\frac{81}{5}\cos^5\theta+C = -(9-x^2)^{\frac{3}{2}}+\frac{1}{15}(9-x^2)^{\frac{5}{2}}+C,$$

so that

$$\int x^3\sqrt{9-x^2}\,dx = -(9-x^2)^{\frac{3}{2}}+\frac{1}{15}(9-x^2)^{\frac{5}{2}}+C.$$

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This last problem was quite admittedly pretty intense, but it was chosen since it illustrates many concepts.

0.3.4. *Partial Fractions.*
Sometimes we wish to integrate rational functions, and usually a u-substitution or other tricks will not work, so we try another method, the method of *Partial Fractions*. The general idea here is to break a rational function that we cannot integrate with the usual tricks down into something that we can.

Example 4. *Compute the following integral:*

$$\int \frac{x-7}{x^2+2x-15}\,dx.$$

Solution. *In Example 1 of Section 0.1 we saw that*

$$\frac{x-7}{x^2+2x-15} = \frac{3}{2x+10} - \frac{1}{2x-6}.$$

Making this substitution into the integral we get:

$$\int \frac{x-7}{x^2+2x-15}\,dx = \int \frac{3}{2x+10} - \frac{1}{2x-6}\,dx = \frac{3}{2}\ln|2x+10| - \frac{1}{2}\ln|2x-6| + C.$$

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Exercises. Evaluate the following integrals:

(1) $\int x(x^2-1)\,dx$

(2) $\int \frac{\cos x}{\sin^2 x}\,dx$

(3) $\int x^3\sqrt{x^2+1}\,dx$ (Hint: You do **not** need trig substitution to do this one.)

(4) $\int x\sqrt{1-x^2}\,dx$

(5) $\int \frac{1}{x^2\sqrt{16x^2-1}}\,dx$

(6) $\int \sec^3\theta\,d\theta$

(7) $\int x^2\cos x\,dx$

(8) $\int e^x \sin x\,dx$

(9) $\int \frac{z^2}{3z+2}\,dz$

(10) $\int \frac{x+1}{x^2-x-12}\,dx$

Furthur Study:

(11) Earlier it was mentioned that the technique of integration by parts is going backwards from the Leibniz rule. In this exercise you will show that.
Let f and g be as in the statement of integration by parts and start with the equation for the Leibniz rule for $f(x)g(x)$:

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

and arrive at the (first) equation for integration by parts:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

by integrating both sides of the equation for the Liebniz rule.

0.4. **Series.**

0.4.1. *Issues of Convergence.*

0.4.2. *Reindexing.*

0.4.3. *Power Series.*

Exercises.

0.5. Complex Numbers and Related Topics.

0.5.1. *Euler's Formula.*

There is a beautiful formula relating e, i, sin, and cos. This formula is known as *Euler's Formula.* Euler's formlua interprets the symbol $e^{i\theta}$.

Theorem 1 (Euler's Formula)**.**

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where θ is given in radians.

Complex analysis interprets $e^{i\theta}$ as more than just a symbol, so we will do the same without justification.

Example 1. *Rewrite the following in the form $a + ib$ where $a, b \in \mathbb{R}$:*

(a) $e^{2\pi i}$
(b) $e^{\frac{\pi}{2}i}$
(c) e^{3i}
(d) e^{ri}

Solution.

(a) $e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1 + 0i = 1$
(b) $e^{\frac{\pi}{2}i} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i = i$
(c) $e^{3i} = \cos 3 + i\sin 3$
(d) $e^{ri} = \cos r + i\sin r$

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A slight expansion on Euler's formula yields a new formula that will be of great use to us in a later chapter. The modification we will use is replacing $i\theta$ with $a+ib$. Then using the usual laws of exponentiation we arrive at a new form of Euler's formula:

$$e^{a+ib} = e^a e^{ib} = e^a(\cos b + i\sin b).$$

Now suppose we wanted to make this a function. This function is $f(x) = e^{(a+ib)x}$. This is still well defined (assuming we are working in the complex numbers) and in fact we can still use Euler's formula to get:

$$f(x) = e^{ax}(\cos bx + i\sin bx).$$

This is ultimately the form of Euler's equation we will be using is later chapters. There are also some other things to note regarding Euler's formula:

$$\frac{1}{2}\left(e^{(a+ib)x} + e^{(a-ib)x}\right) = \frac{1}{2}e^{ax}(\cos bx + i\sin bx) + \frac{1}{2}e^{ax}(\cos bx - i\sin bx) = \frac{1}{2}e^{ax}(2\cos bx) = e^{ax}\cos bx,$$

and

$$\frac{1}{2i}\left(e^{(a+ib)x} - e^{(a-ib)x}\right) = \frac{1}{2i}e^{ax}(\cos bx + i\sin bx) - \frac{1}{2i}e^{ax}(\cos bx - i\sin bx) = \frac{1}{2i}e^{ax}(2i\sin bx) = e^{ax}\sin bx.$$

0.6. Multi-Variable Functions.

0.6.1. *Partial Derivatives.*

Suppose you had a function $f(x, y) = x^3y$ and you wanted to find $\dfrac{df}{dx}$. You would be pretty hard pressed to do this unless you knew how y depended on x. But what if y has no dependence on x, i.e. f is a function of **two** independent variables. Now what?

The answer to this question is that we need to rethink our definition of a derivative. We will now formulate a new definition of the derivative, called the **Partial Derivative**:

Definition 1 (Partial Derivatives)**.** *Suppose that f is a function of two variables, say x and y. Then we define the* **Partial Derivatives** *of f by:*

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h},$$

and

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

$\dfrac{\partial f}{\partial x}$ is called the partial derivative of f with respect to x and $\dfrac{\partial f}{\partial y}$ is called the partial derivative of f with respect to y.

There are many notations used for partial derivatives, for example:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y).$$

If you think about it for a minute, you will see that if f is only a function of one variable, then the definition of partial derivatives above coincides with that of a normal derivative. Another point to note is that we can take partial derivatives of a function of as many independent variables as we want! The definition of a partial derivative may seem a bit scary at first, but here is some good news:
To take a partial derivative with respect to one variable, you just treat all the other variables as constants and differentiate as if it were a function of one variable! Here is a few examples of this:

Example 1. *Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ for the following functions:*

(a) $f(x, y) = x^4y^3 + 8x^2y$
(b) $f(x, y) = \sec(xy)$

Solution.

(a) $\frac{\partial f}{\partial x} = 4x^3y^3 + 16xy$ *and* $\frac{\partial f}{\partial y} = 3x^4y^2 + 8x^2$.
(b) $\frac{\partial f}{\partial x} = y\sec(xy)\tan(xy)$ *and* $\frac{\partial f}{\partial y} = x\sec(xy)\tan(xy)$.

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The fun doesn't end here! We can take higher partial derivatives and even mixed partial derivatives! For example, for a function f of two variables, say x and y, the notation for the second derivatives is: $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, and $\frac{\partial^2 f}{\partial y \partial x}$. The other notations are similar, for example in the subscript notation the cooresponding notations are (in the same order): f_{xx}, f_{yy}, f_{xy}, and f_{yx}. In fact, this brings up a very important theorem involving mixed partial derivatives that will be useful to us in the next chapter.

Theorem 1 (Clairaut's Theorem)**.** *Suppose that f is a function such that $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist and are continuous, then*

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Example 2. *Verify Clairaut's Theorem for the following function:*

$$f(x, y) = xye^xy.$$

Solution. *All we need to do is calculate the mixed partial derivatives and verify that they are equal:*

$$\begin{aligned} f_x &= ye^{xy} + xy^2e^{xy} \\ f_{xy} &= e^{xy} + 3xye^{xy} + x^2y^2e^{xy} \\ f_y &= xe^{xy} + x^2ye^{xy} \\ f_{yx} &= e^{xy} + 3xye^{xy} + x^2y^2e^{xy} \end{aligned}$$

Since $f_{xy} = f_{yx}$ we have verified the theorem.

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0.6.2. *Integration of Multivariable Functions.*

Let's return to the function $f(x,y) = x^3y$. Now suppose instead of differentiating it, we want to integrate it. Does the operation

$$\int x^3 y \, dx$$

even make sense? Again the answer is yes, but we have to slighty modify our definition of integration.

Definition 2. *Suppose that f is a function of two variables, say x and y, and let F_x be a function such that $\frac{\partial F_x}{\partial x} = f$ and F_y be a function such that $\frac{\partial F_y}{\partial y} = f$, then*

$$\int f(x,y)\, dx = F_x(x,y) + g(y)$$

and

$$\int f(x,y)\, dy = F_y(x,y) + h(x)$$

where g is an arbitrary differentiable function of y and h is an arbitrary differentiable function of x.

The functions g and h above take the place of the integration constant since we would like the most general anti-derivative of the function such that when we take the corresponding derivative we get back to the original funcion, i.e.

$$\frac{\partial}{\partial x}\left[F_x(x,y) + g(y)\right] = \frac{\partial}{\partial x}F_x(x,y) + \frac{\partial}{\partial x}g(y) = f(x,y) + 0 = f(x,y).$$

Example 3. *Find $\int f(x,y)\, dx$ and $\int f(x,y)\, dy$ if*

$$f(x,y) = \frac{1}{x+y}.$$

Solution.

$$\int \frac{1}{x+y}\, dx = \ln|x+y| + g(y)$$
$$\int \frac{1}{x+y}\, dx = \ln|x+y| + h(x)$$

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Exercises. Find the first partial derivatives of the following functions:

(1) $z = e^{xy}$

(2) $f(x,y,z) = \frac{1}{x^2+y^2+z^2}$

(3) $k(x,y) = \frac{1}{x} + \frac{1}{y}$

Verify Clairaut's theorem for the following functions:

(4) $f(x,y) = xe^{2y}$

(5) $f(p,q) = \sin(p + \cos q)$

Integrate the following functions with respect to all of its variables:

(6) $f(x,y) = x^2 + y^2$

(7) $g(x,y,z) = xz^2e^{xy}$

(8) $k(r,\theta) = r\cos\theta$

0.7. Miscellaneous Things.

0.7.1. *Slope Fields.*

Recall that the derivative gives the slope of a function at whichever point you evaluate the derivative at. Now suppose that you were just given the equation of the derivative of some function, i.e. something like

$$y' = f(x, y).$$

What does it mean? It tells you the slope of the function $z(x)$, such that $z'(x) = f(x, z(x))$, at the point $(x, z(x))$ in the xy-plane. So how do we use this information? All it seems that we know is that it tells us the slope of some function like z above. The problem is that we don't know, given any point (x, y) in the xy-plane such that $f(x, y)$ is defined, if $z(x) = y$. However, what we can do is assume that there is some function z with $z(x) = y$. Thus we can just plug in any point (x, y) into $f(x, y)$. What this will do is if there is a function z such that $z(x) = y$ and $z'(x) = f(x, z(x))$, you will get a number that is the slope of z at the point (x, y). Obviously it is not feasable to do this for every single point in the xy-plane. Instead we will only plug in a few values, enough to fill in a decent sized grid so that we can get some information about z. We will need some way to represent the number that we get from pulgging in the point on the grid. The way we will do this is by, at each point (x, y) of the grid, draw a small line segment with slope $f(x, y)$ passing though the point. Let's illustrate this through an example:

Example 1. *Draw a slope field for the equation:*

$$y' = 2x$$

in the square $[-2, 2] \times [-2, 2] = \{(x, y) \in \mathbb{R}^2 \mid -2 \leq x \leq 2 \text{ and } -2 \leq y \leq 2\}$.

Solution. *Let's do this by just finding the slopes at the points with integer coordinates:*

$f(-2,2) = -4$	$f(-1,2) = -2$	$f(0,2) = 0$	$f(1,2) = 2$	$f(2,2) = 4$
$f(-2,1) = -4$	$f(-1,1) = -2$	$f(0,1) = 0$	$f(1,1) = 2$	$f(2,1) = 4$
$f(-2,0) = -4$	$f(-1,0) = -2$	$f(0,0) = 0$	$f(1,0) = 2$	$f(2,0) = 4$
$f(-2,-1) = -4$	$f(-1,-1) = -2$	$f(0,-1) = 0$	$f(1,-1) = 2$	$f(2,-1) = 4$
$f(-2,-2) = -4$	$f(-1,-2) = -2$	$f(0,-2) = 0$	$f(1,-2) = 2$	$f(2,-2) = 4$

Now, as described above, we draw a short line segment passing through each of these points of the indicated slope and get:

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At first sight it might not look like these slope fields are too useful, however, observe the following picture which is a slope field for the same equation as above, but with many more sample points:

Do you see anything now? It should look like you can trace these line and get parabolas, and in fact you can! The parabolas that you traced out are called *integral curves* of the equation $y' = 2x$. But why do we get parabolas? Maybe you see why, or maybe not... Either way we will discuss why in the next chapter. One more thing we need to mention here is what to do if $f(x, y)$ is undefined? If $f(x, y)$ is of the form $\frac{c}{0}$, then for the slope, just make it vertical (in fact this pretty much corresponds to having "infinite" slope at that point).

Exercises. Draw a slope field for the following equations and, if you can, trace out a few integral curves. What function do the integral curves look like (if it is possible to identify them)?

(1) $y' = \dfrac{1}{x}$

(2) $y' = \dfrac{x}{y}$

(3) $y' = 3x^2$

(4) $y' = \cos x$

Below each equation is one of its slope fields. Trace out some integral curves in each:

(5) $y' = x^2(1 + y^2)$

(6) $y' = \sin(xy)$

(7) $y' = \dfrac{1}{1 + x^2}$

1. First-Order Differential Equations

1.1. Introduction to Differential Equations.

Differential equations are an extremely useful tool. They can be used to describe anything from a rocketship launching into outer space to the spread of a rumor, from electrical circuts to interest on a loan. With all these applications, one might ask, "What exactly is a differential equation?" Here you go:

Definition 1 (Differential Equation)**.** *A differential equation is an equation containing one or more derivatives of a single unknown function.*

For a function of one variable, in symbols, a differential equation has the form:

$$p(x, y, y', y'', ..., y^{(n)}) = f(x)$$

where p is any function of the indicated inputs, y, the solution of the differential equation, is a function of x, and f is any function of x. Differential equations involving derivatives of a function of one variable are called **Ordinary Differential Equations**, often abbreviated to ODE. Some examples of an ODE are:

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad \nu \in [0, \infty) \tag{1.1}$$

$$y' + y = x^2 \tag{1.2}$$

$$y^{(4)} + 3xy''' + 16\cos(x)y'' + e^x y' + 3y = 4 \tag{1.3}$$

Equation (1.1) is known as the Bessel Equation of order ν.

Notice that there is no restriction on the number of independent variables that the unknown function may have. For example the equation:

$$\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} = 0$$

where u is a function of x and y is a perfectly valid differenital equation. This type of differential equation is known as a **Partial Differential Equation**, often abbreviated to PDE. Some popular PDEs are:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1.4}$$

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{1.5}$$

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}. \tag{1.6}$$

Equation (1.4) is known as the Laplace equation, equation (1.5) is known as the heat equation, and equation (1.6) is known as the wave equation. While PDEs are interesting in their own right, and have copious amounts of applications, in these notes, we will be focusing on ODEs as they are MUCH easier to solve and deal with. From here on out, an ODE will simply be referred to as a differential equation.

In short, a differential equation is an equation with derivatives in it. In the definition of a differential equation there was this mysterious function which the equation contained derivatives of. This function is called the *solution* of the differential equation. Let's see some examples of solutions to differential equations:

First a simple example:

Example 1. *Show that $y = e^{4t}$ is a solution to the differential equation*

$$y' - 4y = 0.$$

Solution. *The way to check that it is a solution is simply to just plug it into the differential equation and verify that both sides of the equation are, in fact, equal. Let's do this:*
First we need to find y':

$$y' = 4e^{4t}.$$

Now just plug y and y' into the equation:

$$y' - 4y = 4e^{4t} - 4(e^{4t}) = 0.$$

Indeed we have that both sides of the equation are equal when we plug in $y = e^{4t}$*, and thus we have verified that* $y = e^{4t}$ *is a solution to the differential equation* $y' - 4y = 0$.

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Now a more complicated example:

Example 2. *Show that* $y = c_1 e^{2x}\sin x + c_2 e^{2x}\cos x$ *is a solution to the differential equation*

$$y'' - 4y' + 5y = 0,$$

where c_1 *and* c_2 *are any two real numbers.*

Solution. *While the equation and the solution may look more complicated, the method remains exactly the same. First we need to find* y' *and* y''*:*

$$\begin{aligned} y' &= c_1(2e^{2x}\sin x + e^{2x}\cos x) + c_2(2e^{2x}\cos x - e^{2x}\sin x) \\ &= 2c_1e^{2x}\sin x + c_1e^{2x}\cos x + 2c_2e^{2x}\cos x - c_2e^{2x}\sin x) \\ &= (2c_1 - c_2)e^{2x}\sin x + (c_1 + 2c_2)e^{2x}\cos x \end{aligned}$$

and

$$\begin{aligned} y'' &= (2c_1 - c_2)(2e^{2x}\sin x + e^{2x}\cos x) + (c_1 + 2c_2)(2e^{2x}\cos x - e^{2x}\sin x) \\ &= (4c_1 - 2c_2)e^{2x}\sin x + (2c_1 - c_2)e^{2x}\cos x + (2c_1 + 4c_2)e^{2x}\cos x - (c_1 + 2c_2)e^{2x}\sin x \\ &= (3c_1 - 4c_2)e^{2x}\sin x + (4c_1 + 3c_2)e^{2x}\cos x \end{aligned}$$

Now that we have both derivatives let's plug them into the differential equation and see if it works out:

$$\begin{aligned} y'' - 4y' + 5y &= (3c_1 - 4c_2)e^{2x}\sin x + (4c_1 + 3c_2)e^{2x}\cos x - 4((2c_1 - c_2)e^{2x}\sin x + (c_1 + 2c_2)e^{2x}\cos x) \\ &\quad + 5(c_1e^{2x}\sin x + c_2e^{2x}\cos x) \\ &= (3c_1 - 4c_2)e^{2x}\sin x + (4c_1 + 3c_2)e^{2x}\cos x - (8c_1 - 4c_2)e^{2x}\sin x - (4c_1 + 8c_2)e^{2x}\cos x + 5c_1e^{2x}\sin x \\ &\quad + 5c_2e^{2x}\cos x \\ &= (3c_1 - 4c_2)e^{2x}\sin x - (8c_1 - 4c_2)e^{2x}\sin x + 5c_1e^{2x}\sin x + (4c_1 + 3c_2)e^{2x}\cos x - (4c_1 + 8c_2)e^{2x}\cos x \\ &\quad + 5c_2e^{2x}\cos x \\ &= (3c_1 - 4c_2 - 8c_1 + 4c_2 + 5c_1)e^{2x}\sin x + (4c_1 + 3c_2 - 4c_1 - 8c_2 + 5c_2)e^{2x}\cos x \\ &= 0e^{2x}\sin x + 0e^{2x}\cos x = 0 \end{aligned}$$

Thus we have verified that $y = c_1e^{2x}\sin x + c_2e^{2x}\cos x$ *is a solution of* $y'' - 4y' + 5y = 0$*, as desired.*

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Definition 2 (Trivial/Nontrivial Solution)**.** *The trivial solution to a differential equation*

$$p(x, y, y', y'', ..., y^{(n)}) = f(x)$$

is the solution $y \equiv 0$ *(i.e.* $y(x) = 0$ *for all* x*). Any other type of solution is called nontrivial.*

Definition 3 (General Solution)**.** *The general solution to the differential equation*

$$p(x, y, y', y'', ..., y^{(n)}) = f(x)$$

is a solution of the form

$$y = y(x, c_1, ..., c_n)$$

where $c_1, ..., c_n$ *are taken to be arbitrary constants.*

For first order differential equations, which is the focus of this chapter, the general solution has the form $y = y(x, c)$.

This last example demonstrates that a differential equation can have a multitude of solutions, in fact infinitely many! What can we do to only get a single solution? Are there ways to put restrictions on the equation in order to get only one solution? The answer is in the affirmative and the way to do this is to impose something called an initial value. Before introducing the concept of an initial value, we must introduce the concept of *order* of a differential equation.

Definition 4 (Order of a Differential Equation)**.** *The order of a differential equation is the order of the highest derivative involved in the differential equation.*

Example 3. *Find the order of the differential equation*

$$y^{(7)} + 25y^{(8)} - 34x^6 y''' + yy' = \sin x^2.$$

Solution. *Since the highest derivative in this equation is $y^{(8)}$, the order of the differential equation is 8.*

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Now we will introduce the concept of an initial value:

Definition 5 (Initial Value)**.** *An initial value for a differential equation of order n, where n is a natural number (i.e. a positive integer), is a specified value for $y^{(k)}(p)$ where $0 \leq k < n$ and p is any point in the domain of the solution.*

An example of an initial value for the equation $y' - 4y = 0$ is $y(0) = 5$. We saw in Example 1 that $y = e^{4t}$ is a solution of $y' - 4y = 0$. In fact any function of the form $y = ce^{4t}$, where c is a real number, is a solution to the equation. To use the initial value we require that if we plug 0 into the function, the output must be 5. So for this equation, take the general form of the solution for y and plug in 0:

$$y(0) = ce^{4(0)} = ce^0 = c$$

thus we see that $y(0) = 5$ is satisfied when $c = 5$, thus we have cut down the number of solutions from infinitely many to one, namely $y = 5e^{4t}$.

In order to garantee that we have cut down the possible solutions for a differential equation from infinitely many to only one, the number of initial values must be one less than the order of the differential equation. Moreover, they must also all be at the same point. More concisely, for a differential equation of order n, the initial values must be of the form:

$$y^{(n-1)}(p) = a_{n-1}, y^{(n-2)}(p) = a_{n-2}, ..., y'(p) = a_1, y(p) = a_0$$

where $a_0, ..., a_{n-1}$ are real numbers.

We have already seen one example of an initial value problem, now let's establish a definition:

Definition 6 (Initial Value Problem)**.** *An initial value problem (often abbreviated IVP) of order n is a differential equation of order n together with n initial values of the form $y^{(n-1)}(p) = a_{n-1}, y^{(n-2)}(p) = a_{n-2}, ..., y'(p) = a_1, y(p) = a_0$. Written more compactly:*

$$g(x, y, y', ..., y^{(n)}) = f(x), y^{(n-1)}(p) = a_{n-1}, y^{(n-2)}(p) = a_{n-2}, ..., y'(p) = a_1, y(p) = a_0.$$

Let's see another example of a solution to an IVP:

Example 4. *Verify that $y = -3e^x + 2e^{3x}$ is a solution to the IVP*

$$y'' - 4y' + 3y = 0, y(0) = -1, y'(0) = 3.$$

Solution. *This is similar to verifying that it is a solution to the differential equation, just with the additional requirement that we check that it also satisfies the initial value. First, let's verify that it satisfies the differential equation. So let's find y' and y'':*

$$y' = -3e^x + 6e^{3x}$$

and

$$y'' = -3e^x + 18e^{3x}$$

and plug them in:

$$y'' - 4y' + 3y = -3e^x + 18e^{3x} - 4(-3e^x + 6e^{3x}) + 3(-3e^x + 2e^{3x}) = (-3 + 12 - 9)e^x + (18 - 24 + 6)e^{3x} = 0.$$

So it satisfies the differential equation, now does it satisfy the initial values?

$$y(0) = -3e^0 + 2e^{3(0)} = -3 + 2 = -1$$

and

$$y'(0) = -3e^0 + 6e^{3(0)} = -3 + 6 = 3.$$

So it also satisfies the initial values, thus $y = -3e^x + 2e^{3x}$ is the solution to the IVP.

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The concept of a slope field is very much related to first order differential equations. Recall that we graph the slope field using the equation for the derivative, which is coincidently (or is it?) a first order differential equation. Remember how, in a slope field, we can trace out curves? These are actually solutions to the differential equation! Often times it is hard to actually find solutions to differential equations, so if finding a closed form solution is impractical, an alternative is to graph the solution curves and try our best to approximate them. There are various ways to do this approximation, some of these methods make up the body of Chapter 8. There are more powerful ways outside of that chapter, and that is the main topic of the field of mathematics called *Numerical Analysis*. Now back to the topic of slope fields. The fact that you can trace out infinitely many of these curves reinforces that a differential equation can have infinitely many solutions. If you consider the most general solution to a differential equation (the one with an arbitrary constant in it), it generates what is called a *family of solutions* to the differential equation. If we specify an initial condition $y(x_0) = y_0$, then we get a specific member of this family of solutions, often called the *solution curve* or the *integral curve* passing through (x_0, y_0).

Exercises.

Verify that the given function is a solution of the differential equation.

(1) $y = \frac{1}{3}x^2 + c\frac{1}{x}$; $xy' + y = x^2$

(2) $y = ce^{-x^2} + \dfrac{1}{2}$; $y' + 2xy = x$

(3) $y = x^{-\frac{1}{2}}(c_1 \sin x + c_2 \sin x) + 4x + 8$; $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 4x^3 + 8x^2 + 3x - 2$

(4) $y = \tan\left(\frac{1}{3}x^3 + c\right)$; $y' = x^2(1 + y^2)$

(5) $y = c_1 \cos 2x + c_2 \sin 2x$; $y'' + 4y = 0$

(6) $y = \sqrt{x}$; $2x^2y'' + 3xy' - y = 0$

Find the order of the given differential equation.

(7) $y'' + 2y'y''' + x = 0$

(8) $y' - y^7 = 0$

(9) $\frac{d^2y}{dx^2}y - \left(\frac{dy}{dx}\right)^2 = 2$

Determine whether the given function is a solution of the initial value problem.

(10) $y = x \cos x$; $y' = \cos x - y \tan x$, $y\left(\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$

(11) $y = \frac{2}{x-2}$; $y' = \frac{-y(y+1)}{x}$, $y(1) = 2$

(12) $y = 4e^x + e^{2x}$; $y'' - 3y' + 2y = 0$, $y(0) = 4$, $y'(0) = 6$

(13) $y = \frac{1}{3}x^2 + x - 1$; $y'' = \dfrac{x^2 - xy' + y + 1}{x^2}$, $y(1) = \frac{1}{3}$, $y'(1) = \frac{5}{3}$

Challenge Problem

(14) Suppose that a function f is a solution of the initial value problem

$$y' = x^2 + y^2, y(1) = 2.$$

Find $f'(1)$, $f''(1)$, and $f'''(1)$.

1.2. **Seperable Differential Equations.**

Suppose we had the differential equation:

$$\frac{dy}{dx} = f(x, y)$$

and that it can be written in the form:

$$p(y)\frac{dy}{dx} = q(x) \tag{1.7}$$

where p and q are some functions of y and x respectively. Suppose that $y = g(x)$ is a solution to this differential equation, then we have that:

$$p[g(x)]g'(x) = q(x).$$

Now let's integrate both sides with respect to x:

$$\int p[g(x)]g'(x)\,dx = \int q(x)\,dx$$

using u-substitution on the left integral with $u = g(x)$ so that $du = g'(x)dx$ we get:

$$\int p(u)\,du = \int q(x)\,dx,$$

which, if we let P and Q be differentiable functions such that $P'(y) = p(y)$ and $Q'(x) = q(x)$ yields, after replacing the u-substitution:

$$P[g(x)] = Q(x) + C,$$

where C is an arbitrary constant. Replacing g with y we have the equation:

$$P(y) = Q(x) + C. \tag{1.8}$$

This says that y is a solution of (1.7) if it satisfies (1.8). Conversely, suppose that y is a function satisfying (1.8). Then taking the derivative of both sides we have:

$$P'(y)\frac{dy}{dx} = Q'(x),$$

or rather

$$p(y)\frac{dy}{dx} = q(x)$$

which is precisely equation (1.7)! This means that y is a solution to (1.7) if and only if it satisfies equation (1.8). This motivates the following definition:

Definition 1 (Separable Equation)**.** *A differential equation*

$$y' = f(x, y),$$

is called separable if it can be written in the form

$$p(y)y' = q(x),$$

or equivalently

$$p(y)dy = q(x)dx.$$

Notice that in equation (1.8) it is not always possible to isolate the y variable and get an *explicit solution* to (1.7), however (1.8) still represents a solution of (1.7), it is called an *implicit solution.* Now let's see some examples of separable differential equations.

Example 1. *Solve the differential equation*

$$y' = -2xy^2.$$

Solution. *Let's start by separating the variables. We can rewrite the equation as*

$$\frac{1}{y^2}dy = -2x\,dx.$$

Next we should integrate both sides of the equation:

$$\int \frac{1}{y^2}\,dy = \int -2x\,dx$$

and we get

$$-\frac{1}{y} = -x^2 + C$$

and thus

$$y = \frac{1}{x^2 + C}$$

is the general solution to our differential equation. However, we are not done yet! Notice that when we separated the variables, we divided by y^2... What if $y^2 = 0$? Then this is a bad move! Thus when we divided by y^2, we implicitly assumed that $y \neq 0$, thus we must check whether $y = 0$ is a solution to the equation. Plugging $y = 0$ into the equation, we see that $y = 0$ is in fact a solution. Since there is no C value we can choose in the general solution that gives $y = 0$, we must include it separately. Thus our complete solution is:

$$y = \frac{1}{x^2 + C}, y \equiv 0$$

◇

Now suppose that, for the example above we include the initial value $y(0) = -1$. To find a solution we plug the initial value into the general solution $y = \dfrac{1}{x^2 + C}$ to solve for C:

$$-1 = \frac{1}{0 + C}$$

which gives that $C = -1$. So our solution to the initial value problem is:

$$y = \frac{1}{x^2 - 1}.$$

There is only one problem with this, a solution to a differential equation has to be a differentiable function, and in order to be differentiable you have to be continuous, and this function is not continuous at $x = \pm 1$!!! To fix this problem, we have to find an interval on which this function is differentiable. $y = \dfrac{1}{x^2 - 1}$ is differentiable on the following intervals: $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. So which one do we choose? By convention we choose the *longest* interval that *contains as many positive numbers as possible*. Thus in this case we will choose $(1, \infty)$. Notice that this is only a convention, so if you chose either of the other two, you would not be wrong.

Now suppose instead that we wanted to start with the initial value $y(2) = 0$. As before, plug this into the general solution:

$$0 = \frac{1}{9 + C}$$

and solving for C we get... WAIT! We CAN'T solve for C! What do we do now? Well remember that $y \equiv 0$ is a solution too, and in fact satisfies this initial value. Thus $y \equiv 0$ is the solution we are looking for.

Another way to handle initial values is the following: Suppose we are given the initial value problem

$$p(y)dy = q(x)dx, y(x_0) = y_0.$$

As usual start by integrating both sides, but this time we are going to incorporate the initial value into the integral:

$$\int_{y_0}^{y} p(y)dy = \int_{x_0}^{x} q(x)dx,$$

then solve for y if possible, otherwise leave it as an implicit solution. It should make sense that the initial value is the lower limit in the integrals since the starting place of the integral is the lower limit.

Example 2. *Solve the differential equation:*

$$y' = 4\frac{y}{x}.$$

Solution. *First start by separating the variables (again, since we are dividing by y we will have to check the solution $y \equiv 0$ separately):*

$$\frac{1}{y}dy = \frac{4}{x}dx.$$

Integrating both sides yields:

$$\ln|y| = 4\ln|x| + C = \ln x^4 + C.$$

Trying to isolate y we get:

$$|y| = e^{\ln x^4 + C} = e^{\ln x^4}e^C = e^C x^4$$

and getting rid of the absolute value on y gives:

$$y = \pm e^C x^4.$$

Since $\pm e^C$ *can attain any value other than* 0, *we might as well write our solution as:*

$$y = Cx^4, C \neq 0.$$

Now that we have done this, let's check whether $y \equiv 0$ *is a solution. Plugging* $y \equiv 0$ *into the equation we get:*

$$0 = 4\frac{0}{x} = 0,$$

thus $y \equiv 0$ *is a solution. Notice that in our equation* $y = Cx^4$, *if we do let* $C = 0$ *we just get* $y = 0$, *which we just verified is a solution, thus we can remove the restriction that* $C \neq 0$ *and write our general solution as:*

$$y = Cx^4.$$

Just one more thing, notice that in our original equation, if we plug in $x = 0$ *we have problems, thus we have to eliminate* $x = 0$ *from the domain of our solution, but that would make it discontinuous, so we only consider our solution on the interval* $(0, \infty)$. *So, at last, our final solution looks like:* $y = Cx^4, x > 0$.

◇

One more example, just for good measure:

Example 3. *Solve the initial value problem*

$$y' = \frac{\sec x \tan x}{5y^4 + 2e^{2y}}, y(0) = 3$$

Solution. *Since* $5y^4 + 2e^{2y} \neq 0$ *for any* y, *we can safely multiply by it to both sides and get:*

$$(5y^4 + 2e^{2y})y' = \sec x \tan x.$$

Integrating both sides we get:

$$y^5 + e^{2y} = \sec x + C.$$

Most likely we won't be able to solve for y *in this case, so it is best to leave it as an implicit solution. Now let's handle the initial value. Plug in the point* $(0, 3)$ *to get:*

$$243 + e^6 = 1 + C \quad \Longrightarrow \quad C = e^6 + 242,$$

thus our solution to the IVP is

$$y^5 + e^{2y} = \sec x + e^6 + 242.$$

◇

Exercises.

Find the explicit general solution of the differential equation. If an explicit solution cannot be found, an implicit solution is acceptable. If there is an initial value, find the solution to the initial value problem.

(1) $y' = xye^x$

(2) $yy' = 4x, y(1) = -3$

(3) $y' = \frac{1+y^2}{1+x^2}, y(2) = 3$ (there actually is an explicit solution to this one!)

(4) $y' = 2y(y-2)$

(5) $y' + y = 6$

(6) $y' = e^x(1-y^2)^{\frac{1}{2}}, y(0) = \frac{1}{2}$

(7) $y' = e^{x+y}$

(8) $y' = 2xy$

(9) $xy' = (1-2x^2)\tan y$

(10) $y' + 2x(y+1) = 0, y(0) = 2$

(11) $y'\sqrt{1-x^2} + \sqrt{1-y^2} = 0$ (this one has an explicit solution too!)

(12) $x + yy' = 1, y(3) = 4$

(13) $2x + 2yy' = 0$

Challenge Problems

(14) Show that an equation of the form $y' = F(ay + bx + c)$, $a \neq 0$, becomes separable under the change of dependent variable $v = ay + bx + k$, where k is any number.

(15) Use exercise (14) to solve the differential equation

$$y' = (y + 4x - 1)^2.$$

(16) Solve the initial value problem

$$y' = e^x(\sin x)(y + 1), y(2) = -1.$$

1.3. **Exact Differential Equations.**

1.3.1. *Exact.*

In this section, we will consider slightly more general differential equations: Exact Differential Equations.

Definition 1 (Exact Differential Equation)**.** *A differential equation*

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

This might seem kind of familiar if you have taken a class in multivariable calculus (specifically here at UCR: Math 10B). Checking whether a differential equation is exact is the exact same process as checking whether a 2-D vector field is conservative. A conservative vector field is a vector field which is the gradient of a scalar function, i.e. suppose that V is a conservative 2-D vector field, then:

$$V = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix},$$

where $f = f(x,y)$ is some function. The way to check whether an arbitrary vector field

$$V = \begin{pmatrix} M(x,y) \\ N(x,y) \end{pmatrix}$$

is conservative is to check if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

The reason this is sufficient is because of Clairaut's theorem which states for a C^2 (all second partial derivatives are continuous) f,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Notice that the condition for conservative is the same as that of being an exact differential equation (at least in appearance). This reason is because they are precisely the same thing. A function $f(x,y)$ is a solution of the exact differential equation $M(x,y)\,dx + N(x,y)\,dy = 0$ if the differential of f (denoted df) is equal to the equation in question, i.e.:

$$df(x,y) := \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y} = 0.$$

Since any C^2 function satisfies Clairaut's theorem, the condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

is sufficient to determine whether a differential equation is exact, because a more technical definition of an exact differential equation is $df = 0$ (where, again, df stands for the differential of f as defined above). You may also see exact differential equations written in one of the following equivalent forms:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0,$$

or

$$M(x,y) + N(x,y)y' = 0.$$

Example 1. *Show that*

$$(3x^2 - 2y^2)dx + (1 - 4xy)dy = 0$$

is exact.

Solution. *In our example $M = 3x^2 - 2y^2$ and $N = 1 - 4xy$. Let's check the exactness equation:*

$$\frac{\partial M}{\partial y} = -4y, \quad \frac{\partial N}{\partial x} = -4y,$$

thus our equation is exact since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

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1.3.2. *Solving Exact Differential Equations.*

Recall that a solution to an exact equation is a function, say f, such that $df = M\,dx + N\,dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}$. So let's start assuming that the equation $M\,dx + N\,dy = 0$ is exact. Then we have the two equations:

$$\frac{\partial f}{\partial x} = M$$

and

$$\frac{\partial f}{\partial y} = N.$$

Let's start with the first equation, and integrate both sides with respect to x. Let $m(x,y)$ be any C^2 function such that $\frac{\partial m}{\partial x} = M$, then:

$$\int \frac{\partial f}{\partial x}(x,y)\,dx = \int M(x,y)\,dx$$

which becomes:

$$f(x,y) = m(x,y) + g(y)$$

where g is an arbitrary function of y. Now let's use our second equation. So take the partial derivative of f with respect to y and set it equal to $N(x,y)$:

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial m}{\partial y}(x,y) + g'(y) = N(x,y).$$

Since our equation is exact we will be able to solve for $g'(y)$ (you are allowed to take this fact on faith). Once we solve for $g'(y)$, integrate it to find $g(y)$. Then we have found our final solution:

$$f(x,y) = m(x,y) + g(y),$$

usually written as:

$$m(x,y) + g(y) = c.$$

Notice that I started by integrating M first. It is perfectly correct, and sometimes easier, to integrate N first; which one you would integrate first just depends on the given equation.

Example 2. *Solve the exact differential equation*

$$3x^2 - 2y^2 + (1 - 4xy)y' = 0.$$

Solution. *We have already shown that this equation is exact in Example 1, now let's find it's solution. Start by integrating M with respect to x and call it $f(x,y)$:*

$$f(x,y) = \int M(x,y)\,dx = \int 3x^2 - 2y^2\,dx = x^3 - 2xy^2 + g(y).$$

Now take the partial derivative of f with respect to y:

$$\frac{\partial f}{\partial y} = -4xy + g'(y).$$

Set $\frac{\partial f}{\partial y} = N$ and solve for $g'(y)$:

$$-4xy + g'(y) = 1 - 4xy,$$

which gives that

$$g'(y) = 1.$$

Integrating both sides we get

$$g(y) = y + c \qquad \textit{(don't forget the integration constant here!!!)},$$

and so our solution is:

$$f(x,y) = x^3 - 2xy^2 + y + c$$

or simply

$$x^3 - 2xy^2 + y = c.$$

◇

Example 3. *Determine whether the equation*

$$(xy^2 + 4x^2y)\,dx + (3x^2y + 4x^3)\,dy = 0$$

is exact, if it is, find the solution.

Solution. *Let's check the exactness condition:*

$$\frac{\partial M}{\partial y} = 2xy + 4x^2, \quad \frac{\partial N}{\partial x} = 6xy + 12x^2.$$

Since the two equations are not equal, the equation is not exact.

◇

1.3.3. *Integrating Factors.*

In the last example we saw an example of a differential equation that is not exact... but that does not mean that it cannot be made exact. If we multiply the function $\mu(x,y) = x^{-1}y$ through the equation, we get:

$$(y^3 + 4xy^2)\,dx + (3xy^2 + 4x^2y)\,dy = 0,$$

which if we check the exactness condition for:

$$\frac{\partial M}{\partial y} = 3y^2 + 8xy, \quad \frac{\partial N}{\partial x} = 3y^2 + 8xy$$

we see that it is now exact!!! The function $\mu(x,y)$ is called an integrating factor for the differential equation.

Definition 2 (Integrating Factor)**.** *A function $\mu(x,y)$ is called an integrating factor for the equation*

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{1.9}$$

if the equation

$$\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0 \tag{1.10}$$

is exact.

Note that Equations (1.9) and (1.10) need not have the same solutions if $\mu(x,y)$ is ever undefined or equal to 0 as muliplication of the equation by such a μ can create singularities or trivialities. Another thing worth noting is that making an equation exact is not the only purpose of an integrating factor.

Integrating factors, in general, are very difficult to find, and while we have many methods for finding them, there is no guarantee that any given method will work on an equation you are trying to solve. In this text we will explore two kinds of integrating factors. First let's find out what we can by just assuming that the Equation (1.9) has an integrating factor $\mu(x,y)$. If it is assumed that $\mu M\,dx + \mu N\,dy = 0$ is exact, then

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

or equivalently

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x,$$

and more usefully

$$\mu(M_y - N_x) = \mu_x N - \mu_y M. \tag{1.11}$$

Suppose that Equation (1.9) has an integrating factor of the form $\mu(x,y) = P(x)Q(y)$. Then $\mu_x(x,y) = P'(x)Q(y)$ and $\mu_y(x,y) = P(x)Q'(y)$. Plugging this into Equation (1.11) we get

$$P(x)Q(y)(M_y - N_x) = P'(x)Q(y)N - P(x)Q'(y)M.$$

Dividing through by $P(x)Q(y)$ we get:

$$M_y - N_x = \frac{P'(x)}{P(x)}N - \frac{Q'(y)}{Q(y)}M \tag{1.12}$$

Now define the functions

$$p(x) = \frac{P'(x)}{P(x)}, \quad \textit{and} \quad q(y) = \frac{Q'(y)}{Q(y)},$$

and plug them into (1.12) to get:

$$M_y - N_x = p(x)N - q(y)M. \tag{1.13}$$

Thus we have a condition on when $\mu(x,y)$ is an integrating factor. However, (1.13) is more useful than you might think! Let's suppose that we have two functions p and q satisfying (1.13). Then by comparison with (1.12), we see that

$$p(x) = \frac{P'(x)}{P(x)}, \quad \textit{and} \quad q(y) = \frac{Q'(y)}{Q(y)}.$$

Thus by integrating both sides of the equations we find that:

$$P(x) = e^{\int p(x)\,dx}, \quad \textit{and} \quad Q(y) = e^{\int q(y)\,dy}.$$

While this is a good result, there is no guarantee that we can find such functions $p(x)$ and $q(y)$ satisfying (1.13). Here are some conditions with which we can find p and q:

Theorem 1. *Consider the differential equation*

$$M\,dx + N\,dy = 0.$$

(a) *If* $\dfrac{M_y - N_x}{N}$ *is independent of* y *(does not contain the variable* y*), define*

$$p(x) = \frac{M_y - N_x}{N}.$$

Then

$$\mu(x) = e^{\int^x p(x)\,dx}$$

is an integrating factor for $M\,dx + N\,dy = 0$.

(b) *If* $\dfrac{N_x - M_y}{M}$ *is independent of* x *(does not contain the variable* x*), define*

$$q(y) = \frac{N_x - M_y}{M}.$$

Then

$$\mu(y) = e^{\int^y q(y)\,dy}$$

is an integrating factor for $M\,dx + N\,dy = 0$.

There are some differential equations where it will be easier to apply part (a) of the theorem (it can even be impossible to apply (b) rather than just more difficult), and vice versa; there are cases where it is just as easy to apply either one; and there are cases where it is impossible to apply either one. Just a reminder, there is no guarantee that a differential equation has an integrating factor that makes it exact.

Example 4. *Find an integrating factor for the equation*

$$(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)dx + (3x^2y^2 + 4y)dy = 0$$

and find the solution.

Solution. *Let's try to apply Theorem 1. First let's find* M_y *and* N_x*:*

$$M_y = 6xy^2 - 6x^3y^2 - 8xy$$

and

$$N_x = 6xy^2.$$

Thus the equation is not exact, so it is necessary to find an integrating factor. Well:

$$M_y - N_x = -6x^3y^2 - 8xy$$

and

$$N = 3x^2y^2 + 4y$$

so

$$\frac{M_y - N_x}{N} = \frac{-6x^3y^2 - 8xy}{3x^2y^2 + 4y} = \frac{-2x(3x^3y^2 + 4y)}{3x^3y^2 + 4y} = -2x$$

which is independent of y*, so letting* $p(x) = -2x$ *and applying Theorem 1.a we get*

$$\mu(x) = e^{\int p(x)\,dx} = e^{\int -2x\,dx} = e^{-x^2}$$

as our integrating factor. Now multiply through the differential equation by μ *to get:*

$$e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)dx + e^{-x^2}(3x^2y^2 + 4y)dy = 0.$$

In this case it will be much easier to compute the integral $\int \mu N\,dy$ *so let's do that:*

$$f = \int \mu N\,dy = \int e^{-x^2}(3x^2y^2 + 4y)dy = e^{-x^2}\int (3x^2y^2 + 4y)dy = e^{-x^2}(x^2y^3 + 2y^2) + g(x).$$

Now find $\dfrac{\partial f}{\partial x}$ *and set it equal to* μM*:*

$$\frac{\partial f}{\partial x} = -2xe^{x^2}(x^2y^3+2y^2)+e^{-x^2}(2xy^3)+g'(x) = e^{-x^2}(-2x^3y^3-4xy^2+2xy^3)+g'(x) = \mu M = e^{-x^2}(2xy^3-2x^3y^3-4xy^2+2x)$$

Solving for $g'(x)$ yields

$$g'(x) = 2xe^{-x^2}$$

and thus

$$g(x) = -e^{-x^2}$$

so that our solution is

$$e^{-x^2}(x^2y^3 + 2y^2 - 1) = c.$$

◇

Remark. *It is sometimes convineint for differential equations with M and N both polynomials (in two variables) to first check for an integrating factor of the form $\mu(x,y) = x^m y^n$. This can often times be less work than Theorem 1, but is much less general.*

Remark. *Theorem 1 does not always work. If you find that it does not work, an alternative that you can (and should) try it to try to find functions $p(x)$ and $q(y)$ that satisfy Equation (1.13) and then find $\mu(x,y) = P(x)Q(y)$ using the method outlined following Equation (1.13). An example of this type of problem is Exercise 7.*

Exercises.

Check if the equation is exact, if it is, solve it. If there is an initial value, find the particular solution satsifying the initial value.

(1) $(3x^2y^2 - 4xy)y' = 2y^2 - 2xy^3$

(2) $(x + y^2)\dfrac{dy}{dx} + 2x^2 - y = 0$

(3) $(4x^3y^2 - 6x^2y - 2x - 3)dx + (2x^4y - 2x^3)dy = 0, y(1) = 3$

(4) $(x^2 - y)dy + (2x^3 + 2xy)dx = 0$

(5) $(y^{-3} - y^{-2}\sin x)y' + y^{-1}\cos x = 0$

(6) $(2x - 1)(y - 1)dx + (x + 2)(x - 3)dy = 0, y(1) = -1$

Solve the differential equation if possible, using an integrating factor if necessary.

(7) $(3xy + 6y^2)dx + (2x^2 + 9xy)dy = 0$

(8) $y\,dx - x\,dy = 0$

(9) $(2xy + y^2)dx + (2xy + x^2 - 2x^2y^2 - 2xy^3)dy = 0$

(10) $\cos x \cos y\,dx + (\sin x \cos y - \sin x \sin y + y)dy = 0$

(11) $2y\,dx + 3(x^2 + x^2y^3)dy = 0$

(12) $(1 - xy)y' + y^2 + 3xy^3 = 0$

(13) $(x^2 + xy^2)y' - 3xy + 2y^3 = 0$

Challenge Problems:

(14) Show that a separable equation is exact.

(15) Let $P(x) = \int p(x)\,dx$. Show that $e^{P(x)}$ is an integrating factor for the linear equation

$$y' + p(x)y = q(x).$$

(16) Suppose that a, b, c, and d are constants such that $ad - bc \neq 0$, and let m and n be arbitrary real numbers. Show that

$$(ax^my + by^{n+1})dx + (cx^{m+1} + dxy^n)dy = 0$$

has an integrating factor $\mu(x,y) = x^\alpha y^\beta$.

1.4. **Linear Differential Equations.**

In Section 1.1 we learned how to classify differential equations in terms of *order*. In this section we will learn another way. The classification is by linearity.

Definition 1 (Linear Differential Equation)**.** *An n^{th} order differential equation is said to be linear if it is of the form*

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

where $a_i(x)$, $0 \le i \le n$, and $f(x)$ are given functions of the independent variable x, and it is assumed that $a_n(x) \not\equiv 0$.

In this chapter, we will be concerned only with first order linear differential equations. Instead of using the above notation, we will commonly write first order linear differential equations in the form

$$y' + p(x)y = q(x). \tag{1.14}$$

This is exactly the same as the definition above, except that we divided through by $a_1(x)$. From here we can classify linear differential equations further, using the terms *homogeneous* and *inhomogeneous*.

Definition 2 (Homogeneous/Nonhomogeneous Linear Differential Equations)**.** *The linear differential equation*

$$y' + p(x)y = q(x)$$

is called homogeneous if $q(x) \equiv 0$ and nonhomogeneous otherwise.

For a nonhomogeneous linear differential equation $y' + p(x)y = q(x)$, the *corresponding homogeneous equation* is

$$y' + p(x)y = 0. \tag{1.15}$$

Something worth noting is that a homogeneous linear equation is always separable. This being so, I will skip how to solve homogeneous linear differential equations and just start with nonhomogeneous linear differential equations. This leads us to the next topic.

1.4.1. *Variation of Parameters.*

Suppose we had a nonhomogeneous linear equation of the form (1.14). Let's start by solving the corresponding homogeneous equation (1.15). Let y_1 be a solution (a specific one, not the general one) to (1.15), i.e. $y_1' + p(x)y_1 = 0$. Now we will search for a solution of (1.14) of the form $y = uy_1$, where u is a function that is to be determined. Start by plugging in $y = uy_1$ into (1.14) to get:

$$u'y_1 + uy_1' + p(x)uy_1 = q(x).$$

Factoring out u from the second and third terms, since y_1 is a solution to (1.15) we have:

$$u'y_1 + u(y_1' + p(x)y_1) = u'y_1 + 0 = u'y_1 = q(x),$$

thus

$$u' = \frac{q(x)}{y_1}.$$

Now integrate both sides of the above equation to get

$$u = \int \frac{q(x)}{y_1}dx.$$

So our final solution is of the form:

$$y = y_1 \int \frac{q(x)}{y_1}dx.$$

The method we used here is called *Variation of Parameters* and will be especially useful in the next chapter when we study second order linear differential equations.

Example 1. *Solve the initial value problem*

$$xy' - 2y = -x^2, y(1) = 1.$$

Solution. *The first thing we should do is make it look like our definition of a linear equation, so divide through by x to get:*

$$y' - \frac{2}{x}y = -x.$$

Now let's solve the corresponding homogeneous equation:

$$y' - \frac{2}{x}y = 0.$$

Separating the variables we get:

$$\frac{1}{y}dy = \frac{2}{x}dx$$

and integrating both sides:

$$\ln|y| = 2\ln|x| + c$$

lastly, solving for y we get:

$$y = cx^2$$

as our general solution to the homogeneous equation (again we did divide by y, so we should check that $y = 0$ is a solution, and in fact it is, so we may allow c above to be any real number). Now let's find the general solution to the nonhomogeneous equation. Let $y_1 = x^2$ and following the method outlined above, let

$$y = uy_1 = ux^2.$$

Then

$$y' = u'y_1 + uy_1' = u'x^2 + 2ux,$$

and plugging it in we get:

$$y' - \frac{2}{x}y = u'y_1 + uy_1' - \frac{2}{x}uy_1 = u'y_1 = u'x^2 = -x$$

so that

$$u' = -x^{-1}$$

and integrating both sides gives:

$$u = -\ln|x| + c.$$

Thus the general solution to our problem is

$$y = (c - \ln|x|)x^2$$

which we can rewrite as

$$y = x^2(c - \ln x)$$

if we assume that the domain of the solution is $(0, \infty)$, which we will for convenience. Now for the initial value, plug in the point $(1, 1)$:

$$1 = 1^2(c - \ln 1) = c - 0 = c$$

thus $c = 1$ and our particular solution to the initial value problem is

$$y = x^2(1 - \ln x).$$

◇

1.4.2. *Integrating Factors.*

An alternate method to variation of parameters for first order linear equations is using an integrating factor. This is actually the same integrating factor as in the previous section, but more specialized. The purpose of the integrating factor in this section is to rewrite equation (1.14) in the more compact form $(\mu y)' = \mu q(x)$ in which y can easily be solved for.

Theorem 1 (Integrating Factor for First Order Linear Differential Equations)**.** *Suppose we had the differential equation* (1.14)*. Let*

$$\mu(x) = e^{\int^x p(s)ds},$$

(where $\int^x p(s)ds$ means the antiderivative of $p(x)$ with no integration constant, or setting the integration constant equal to 0*) then $\mu(x)$ is an integrating factor for* (1.14)*, and*

$$y = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx$$

is a solution to (1.14)*.*

Proof. Suppose that $\mu(x)$ is an integrating factor for (1.14), then

$$\mu(x)y' + \mu(x)p(x)y = (\mu(x)y)' = \mu'(x)y + \mu(x)y'$$

which simplifies to

$$\mu(x)p(x) = \mu'(x).$$

Solving for $\mu(x)$ we find that

$$\mu(x) = e^{\int p(x)\,dx}$$

and since the integrating factor should not have an arbitrary constant in it, choose

$$\mu(x) = e^{\int^x p(s)\,ds}.$$

So we have found $\mu(x)$ and it fits with the statement of the theorem, now let's solve for y. Integrate both sides of the equation

$$(\mu(x)y)' = \mu(x)q(x)$$

to get

$$\mu(x)y = \int \mu(x)q(x)$$

and dividing by $\mu(x)$ we get:

$$y = \frac{1}{\mu(x)} \int \mu(x)q(x)\,dx$$

as desired. □

This method, while less powerful than variation of parameters in general, is much more efficient to apply since we already know the form of $\mu(x)$ without having to solve the homogeneous part of the linear equation first. Let's see this method in action with an example.

Example 2. *Solve the differential equation*

$$x(\ln x)y' + y = 2\ln x.$$

Solution. *First rewrite it in the proper form:*

$$y' + \frac{1}{x\ln x}y = \frac{2}{x}.$$

Now find the integrating factor, using the substitution $u = \ln x$:

$$\int^x \frac{1}{x\ln x}dx = \int^u \frac{1}{u}du = \ln|u| = \ln|\ln x|$$

so that

$$\mu(x) = e^{\ln|\ln x|} = |\ln x|$$

which we will choose to be

$$\mu(x) = \ln(x)$$

by restricting $x > 1$. Now simply plug $\mu(x)$ and $q(x)$ into the formula for y, and use the substitution $v = \ln x$:

$$y = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = \frac{1}{\ln x}\int \frac{2\ln x}{x}dx = \frac{1}{\ln x}\int 2v\,dv = \frac{1}{\ln x}[(\ln x)^2 + c].$$

So our general solution is

$$y = \ln x + \frac{c}{\ln x}.$$

◇

Exercises.

Solve the differential equation. If there is an initial value, find the particular solution satisfying the initial value.

(1) $xy' + 2y = 4x^2, y(1) = 4$

(2) $y' - 2xy = 1, y(a) = b$

(3) $y' + 2xy = 2x$

(4) $y' + (\sec x \tan x)y = 0$

(5) $y' + (\cot x)y = 3\cos x \sin x$

(6) $y' + (\cos x)y = \cos x, y(\pi) = 0$

(7) $y' + \frac{1}{x}y = \frac{7}{x^2} + 3$

(8) $y' = \frac{1}{x^2 + 1}$

(9) $y' + 2xy = x^2, y(0) = 3$

(10) $xy' + (x+1)y = e^{x^2}$

Challenge Problems:

(11) Show that the method of variation of parameters and the integrating factor method give the same answer for first order linear equations.

(12) Assume that all functions in this exercise have the same domain.

(a) Prove: If y_1 and y_2 are solutions of

$$y' + p(x)y = q_1(x)$$

and

$$y' + p(x)y = q_2(x)$$

respectively, and c_1 and c_2 are arbitrary constants, then $y = c_1y_1 + c_2y_2$ is a solution of

$$y' + p(x)y = q_1(x) + q_2(x).$$

(This is known as the *principle of superposition*.)

(b) Use (a) to show that if y_1 and y_2 are solutions of the nonhomogeneous equation

$$y' + p(x)y = q(x),$$

then $y_1 - y_2$ is a solution of the homogeneous equation

$$y' + p(x)y = 0.$$

(This is known as *uniqueness of nonhomogeneous solutions*.)

(c) Use (a) to show that if y_1 is a solution of $y' + p(x)y = q(x)$ and y_2 is a solution of $y' + p(x)y = 0$, then $y_1 + y_2$ is a solution of $y' + p(x)y = q(x)$. (This shows that the nonhomogeneous solution is independent of the homogeneous solution.)

(13) Some nonlinear equations can be transformed into linear equations by changing the dependent variable. Show that if

$$g'(y)y' + p(x)g(y) = q(x),$$

were y is a function of x and g is a function of y, then the new dependent variable $z = g(y)$ satisfies the linear equation

$$z' + p(x)z = q(x).$$

(14) Use the method outlined in Exercise 13 to solve the following equation:

$$\frac{1}{1+y^2}y' + \frac{2}{x}\arctan y = \frac{2}{x}.$$

(15) Show that a homogeneous linear differential equation is always separable.

1.5. Bernoulli Equations.

In this section we will study a special type of nonlinear first order differential equation that can always be transformed into a linear equation with a change of variable: the *Bernoulli Equation*.

Definition 1 (Bernoulli Equation)**.** *A Bernoulli Equation is a differential equation of the form*

$$y' + p(x)y = q(x)y^n \tag{1.16}$$

where n is any number other than 0 or 1.

Theorem 1. *Making the substitution $u = y^{1-n}$ in the Bernoulli equation (1.16) yields the linear equation*

$$\frac{1}{1-n}u' + p(x)u = q(x).$$

Proof. Start by finding u':

$$u' = (1-n)y^{-n}y'.$$

Now divide equation (1.16) by y^n to get:

$$y^{-n}y' + p(x)y^{1-n} = q(x).$$

Making the proper substitutions we see that

$$\frac{1}{1-n}u' + p(x)u = q(x)$$

as desired. □

Example 1. *Solve the Bernoulli equation*

$$y' + \frac{3}{x}y = x^2y^2, x > 0$$

Solution. *By the look of the equation we can tell that $n = 2$, so it is indeed a Bernoulli equation. We can either use the formula to transform the equation into a linear one, or do it step by step, which is what we will do in this example. Start by dividing the equation by y^2:*

$$y^{-2}y' + \frac{3}{x}y^{-1} = x^2.$$

Now make the substitution $u = y^{-1}$, $u' = -y^{-2}y'$:

$$-u' + \frac{3}{x}u = x^2$$

or equivalently

$$u' - \frac{3}{x}u = -x^2.$$

An integrating factor is:

$$\mu(x) = e^{\int^x \frac{-3}{x}ds} = e^{\ln x^{-3}} = x^{-3}$$

and thus the general solution with respect to u is

$$u = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx = x^3\int \frac{-1}{x}dx = x^3(c - \ln x).$$

Now since, $y = u^{-1}$ we have our general solution is

$$y = x^{-3}(x - \ln x)^{-1}.$$

◇

Exercises.

Solve the differential equation:

(1) $xy' + y + x^2y^2e^x = 0$

(2) $xyy' = y^2 - x^2$

(3) $xy' - (3x+6)y = -9xe^{-x}y^{\frac{4}{3}}$

(4) $(1+x^2)y' + 2xy = \dfrac{1}{(1+x^2)y}$

(5) $x^2y' + 2y = 2e^{\frac{1}{x}}y^{\frac{1}{2}}$

Challenge Problems:

(6) An equation of the form

$$\frac{dy}{dx} = p(x)y^2 + q(x)y + r(x)$$

is called a *Riccati equation*.

(a) If $p(x) \equiv 0$ show that the equation is linear. If $r(x) \equiv 0$ show that it is a Bernoulli equation.

(b) If $y = y_1(x)$ is some particular solution, show that the change of variable $y = y_1(x) + \frac{1}{u}$ leads to a linear equation in u.

(7) Use Exercise 6 to solve the equation. A particular solution is given.

$$y' = y^2 + 2xy + (x^2 - 1); y_1 = -x.$$

1.6. **Homogeneous Differential Equations.**

In the last section we studied a class of nonlinear differential equations that become linear after a change of variable. In this section we will study a class of differential equations that become separable after a change of variable: the *Homogeneous Equation*.

Definition 1. *A homogeneous differential equation is a first order differential equation that can be written in the form*

$$y' = f\left(\frac{y}{x}\right). \tag{1.17}$$

Please do not get this confused with the homogeneous equations of Section 1.4. It is indeed unfortunate that these equations have the same name.

Anyway, a homogeneous equation can be made separable under the change of variable $v = \frac{y}{x}$. Observe: If we let $v = \frac{y}{x}$ then $y = vx$ and $y' = v'x + v$. Plugging this into (1.17) we get:

$$v'x + v = f(v)$$

which separates as follows:

$$\frac{1}{f(v) - v} dv = \frac{1}{x} dx.$$

Example 1. *Solve the homogeneous equation*

$$x^2 y' = xy - y^2.$$

Solution. *First let's try our best to get it in the form of* (1.17)*. So to get y' by itself, first divide by x^2:*

$$y' = \frac{y}{x} - \frac{y^2}{x^2} = \frac{y}{x} - \left(\frac{y}{x}\right)^2.$$

Now let $v = \frac{y}{x}$, then substitution gives:

$$v'x + v = v - v^2.$$

This separates into

$$\frac{1}{v^2} dv = -\frac{1}{x} dx.$$

Integration of both sides gives:

$$-\frac{1}{v} = -\ln|x| + c$$

and solving for v:

$$v = \frac{1}{\ln|x| + c}.$$

Replacing v with $\frac{y}{x}$ we get our general solution:

$$y = \frac{x}{\ln|x| + c}.$$

◇

In general, it is hard to see a priori whether or not a differential equation is homogeneous. We will now describe a method to tell whether a given equation is homogeneous. First we need a definition, which is yet another use of the word homogeneous.

Definition 2 (Homogeneous of Degree m)**.** *A function g of two variables is said to homogeneous of degree m if*

$$g(tx, ty) = t^m g(x, y).$$

Theorem 1. *The differential equation*

$$N(x, y)y' = M(x, y)$$

is homogeneous if both M and N are homogeneous of the same degree.

Proof. Suppose that M and N are homogeneous of the same degree, say m. Rewrite the differential equation in the form

$$y' = \frac{M(x, y)}{N(x, y)}.$$

Since M and N are homogeneous of degree m we have:

$$M(x, y) = t^{-m} M(tx, ty), \quad \textit{and} \quad N(x, y) = t^{-m} N(tx, ty).$$

Let $t = \dfrac{1}{x}$ and plug it into the above two equations to get:

$$M(x,y) = x^m M\left(1, \frac{y}{x}\right) \quad \textit{and} \quad N(x,y) = x^m N\left(1, \frac{y}{x}\right).$$

Now plugging the above two equations into our differential equation gives:

$$y' = \frac{M(x,y)}{N(x,y)} = \frac{x^m M\left(1, \frac{y}{x}\right)}{x^m N\left(1, \frac{y}{x}\right)} = \frac{M\left(1, \frac{y}{x}\right)}{N\left(1, \frac{y}{x}\right)}$$

which obviously only depends on $\frac{y}{x}$ since 1 is a constant, thus it is homogeneous since it is of the form (1.17). □

Example 2. *Determine whether the following equation is homogeneous*

$$y' = \frac{y^3 + 2xy^2 + x^2y + x^3}{x(y+x)^2}.$$

Solution. *By our theorem* $N(x,y) \equiv 1$ *and* $M(x,y) = \dfrac{y^3 + 2xy^2 + x^2y + x^3}{x(y+x)^2}$. *Clearly* N *is homogeneous of degree* 0 *since*

$$N(tx,ty) = 1 = t^0 \cdot 1 = t^0 N(x,y).$$

So we need M *to be homogeneous of degree* 0 *as well:*

$$\begin{aligned} M(tx,ty) &= \frac{(ty)^3 + 2(tx)(ty)^2 + (tx)^2(ty) + (tx)^3}{(tx)[(ty)+(tx)]^2} = \frac{t^3y^3 + 2t^3xy^2 + t^3x^2y + t^3x^3}{tx[t^2(y+x)^2]} \\ &= \frac{t^3(y^3 + 2xy^2 + x^2y + x^3)}{t^3x(y+x)^2} = t^0 \frac{y^3 + 2xy^2 + x^2y + x^3}{x(y+x)^2} = t^0 M(x,y). \end{aligned}$$

Thus our equation is homogeneous.

◇

Exercises.

Determine whether the equation is homogeneous, if it is, solve it. If there is an initial value, find the particular solution satsifying the initial value.

(1) $y' = \dfrac{y+x}{x}$

(2) $y' = \dfrac{xy + y^2}{x^2}, y(-1) = 2$

(3) $xy' = y - xe^{\frac{y}{x}}$

(4) $xy' - y = \sqrt{x^2 + y^2}$

(5) $xy' - y = 2y(\ln y - \ln x)$

(6) $(x^2 + y^2)\dfrac{dy}{dx} = 5y$

(7) $y' = \dfrac{y^2 - 3xy - 5x^2}{x^2}, y(1) = -1$

(8) $(x^3 + x^2y + xy^2)y' = xy^2 + 2y^3$

(9) $(y - 2x)y' = y$

1.7. **Existence and Uniqueness of Solutions to Differential Equations.**

Exercises.

1.8. Additional Exercises.

Solve the differential equation if possible. If there is an initial value, find the particular solution that satisfies the initial value.

(1) $x\,dx + y\,dy = 0$

(2) $dx + \frac{1}{y^4}dy = 0$

(3) $\sin x\,dx + y\,dy = 0, y(0) = -2$

(4) $\frac{1}{x}dx - \frac{1}{y}dy = 0$

(5) $(t^2+1)dt + (y^2+y)dy = 0$

(6) $(x^2+1)dx + \frac{1}{y}dy = 0, y(-1) = 1$

(7) $\frac{4}{t}dt - \frac{y-3}{y}dy = 0$

(8) $dx - \frac{1}{1+y^2}dy = 0$

(9) $xe^{x^2}dx + (y^5-1)dy = 0, y(0) = 0$

(10) $y' = \frac{y}{x^2}$

(11) $y' = \frac{xe^x}{2y}$

(12) $y' = \frac{x^2y - y}{y+1}$

(13) $\frac{dx}{dt} = \frac{x}{t}$

(14) $\frac{dx}{dt} = 8 - 3x, x(0) = 4$

Determine whether the given equation is homogeneous, and if so, solve it.

(15) $y' = \frac{x^2+2y^2}{xy}$

(16) $xy' = y - x$

(17) $(xy + (xy^2)^{\frac{1}{3}})y' = y^2$

(18) $y' = \frac{y}{x+\sqrt{xy}}$

Determine whether or not the given equation is exact. If it is, solve it, if not, try to find an integrating factor. If you cannot find an integrating factor by the methods given above, state so. If there is an initial value, find the particular solution satsifying the initial value.

(19) $(xy+1)dx + (xy-1)dy = 0$

(20) $(y+2xy^3)dx + (1+3x^2y^2+x)dy = 0, y(1) = -5$

(21) $(y+1)dx - xdy = 0$

(22) $e^{x^3}(3x^2y - x^2)dx + e^{x^3}dy = 0$

(23) $-\frac{y^2}{t^2}dt + \frac{2y}{t}dy = 0, y(2) = -2$

(24) $(y + x^3y^3)dx + x\,dy = 0$

(25) $(y\sin x + xy\cos x)dx + (x\sin x + 1)dy = 0$

(26) $-\dfrac{2y}{t^3}dt + \dfrac{1}{t^2} = 0, y(2) = -2$

(27) $(y + x^4y^2)dx + x\,dy = 0$

(28) $y^2\,dt + t^2\,dy = 0$

(29) $(t^2 - x)dt - t\,dx = 0, x(1) = 5$

(30) $2xy\,dx + y^2\,dy = 0$

(31) $\sin t\cos x\,dt - \sin x\cos t\,dx = 0$

(32) $y\,dx + 3x\,dy = 0$

(33) $\left(2xy^2 + \dfrac{x}{y^2}\right)dx + 4x^2y\,dy = 0$

Solve the differential equation. If there is an initial value, find the particular solution satisfying the initial value.

(34) $\dfrac{dy}{dx} + 5y = 0$

(35) $y' + y = y^2$

(36) $\dfrac{dy}{dx} + 2xy = 0$

(37) $y' + \dfrac{2}{x}y = x, y(1) = 0$

(38) $y' + \dfrac{1}{x}y = 0$

(39) $xy' + y = xy^3$

(40) $y' + \dfrac{2}{x}y = 0$

(41) $y' + 6xy = 0, y(\pi) = 5$

(42) $y' - 7y = e^x$

(43) $y' + y = y^2e^x$

(44) $y' = \cos x$

(45) $y' + \dfrac{2}{x}y = -x^9y^5, y(-1) = 2$

(46) $y' - \dfrac{3}{x^2}y = \dfrac{1}{x^2}$

(47) $y' + xy = 6x\sqrt{y}$

(48) $\dfrac{dy}{dx} + 50y = 0$

(49) $\dfrac{dv}{dt} + 2v = 32, v(0) = 0$

(50) $\dfrac{dp}{dt} - \dfrac{1}{t}p = t^2 + 3t - 2$

(51) $y' + y = y^{-2}$

(52) $\dfrac{dQ}{dt} + \dfrac{2}{20-t}Q = 4$

(53) $\dfrac{dN}{dt} + \dfrac{1}{t}N = t, N(2) = 8$

Determine (a) the order, (b) the unknown function, and (c) the independent variable for each of the given differential equations.

(54) $(y'')^2 - 3yy' + xy = 0$

(55) $y^{(4)} + xy''' + x^2y'' - xy' + \sin y = 0$

(56) $\dfrac{d^n x}{dy^n} = y^2 + 1$

(57) $\left(\dfrac{d^2y}{dx^2}\right)^{\frac{3}{2}} + y = x$

(58) $\dfrac{d^7 b}{dp^7} = 3p$

(59) $\left(\dfrac{db}{dp}\right)^7 = 3p$

Determine if the given functions are solutions to the given differential equation.

(60) $y' - 5y = 0$
- (a) $y = 5$
- (b) $y = 5x$
- (c) $y = x^5$
- (d) $y = e^{5x}$
- (e) $y = 2e^{5x}$
- (f) $y = 5e^{2x}$

(61) $y' - 2ty = t$
- (a) $y = 2$
- (b) $y = -\dfrac{1}{2}$
- (c) $y = e^{t^2}$
- (d) $y = e^{t^2} - \dfrac{1}{2}$
- (e) $y = -7e^{t^2} - \dfrac{1}{2}$

(62) $\dfrac{dy}{dx} = \dfrac{2y^4 + x^4}{xy^3}$
- (a) $y = x$
- (b) $y = x^8 - x^4$
- (c) $y = \sqrt{x^8 - x^4}$
- (d) $y = (x^8 - x^4)^{\frac{1}{4}}$

(63) $y'' - xy' + y = 0$
- (a) $y = x^2$
- (b) $y = x$
- (c) $y = 1 - x^2$
- (d) $y = 2x^2 - 2$
- (e) $y = 0$

Find the constant c such that the given function satisfies the given initial values.

(64) $x(t) = ce^{2t}$
- (a) $x(0) = 0$
- (b) $x(0) = 1$
- (c) $x(1) = 1$

(d) $x(2) = -3$

(65) $x(t) = c(1 - x^2)$
(a) $y(0) = 1$
(b) $y(1) = 0$
(c) $y(2) = 1$
(d) $y(1) = 2$

Find the constants c_1 and c_2 such that the given function satisfies the given initial value(s).

(66) $y(x) = c_1 \sin x + c_2 \cos x$
(a) $y(0) = 1, y'(0) = 2$
(b) $y(0) = 2, y'(0) = 1$
(c) $y\left(\frac{\pi}{2}\right) = 1, y'\left(\frac{\pi}{2}\right) = 2$
(d) $y(0) = 0, y'(0) = 0$

(67) $y(x) = c_1 e^x + c_2 e^{-x} + 4 \sin x; y(0) = 1, y'(0) = -1$

(68) $y(x) = c_1 e^x + c_2 e^{2x} + 3e^{3x}; y(0) = 0, y'(0) = 0$

(69) $y(x) = c_1 e^x + c_2 x e^x + x^2 e^x; y(1) = 1, y'(1) = -1$

(70) $y(x) = c_1 \sin x + c_2 \cos x + 1; y(\pi) = 0, y'(\pi) = -1$

2. Second-Order Differential Equations

2.1. Constant Coefficient Homogeneous Linear Equations.

Second order differential equations have historically been studied the most due to the fact that they are very applicable. For example if you are given the position function, $x(t)$ of a moving body, it's acceleration is $x''(t)$. Another example is Newton's law: $F = ma$, which again, if you are given the position function for the body attached to the spring, you get: $F(t) = mx''(t)$. One more example is an adaption of Kirchoff's law for an RLC circuit: $LI' + RI + \frac{1}{C}Q = E(t)$, where L is inductance, I is current, R is resistance, C is capacitance, Q is charge on the capacitance, and E is the applied voltage. This becomes a second order differential equation for the current I by taking the time derivative of both sides: $LI'' + RI' + \frac{1}{C}I = E'$.

Definition 1 (Second Order Linear Differential Equation)**.** *A second order differenital equation is said to be linear if it takes the form*

$$p(x)y'' + q(x)y' + r(x)y = f(x).$$

It is called homogeneous if $f(x) \equiv 0$ and nonhomogeneous otherwise.

In this section we will assume that $p(x)$, $q(x)$, and $r(x)$ are constant functions and that $f(x) \equiv 0$. Thus our equations will take the form

$$ay'' + by' + cy = 0, \tag{2.1}$$

with $a \neq 0$. Clearly the function $y \equiv 0$ is a solution, and it will be called the *trivial* solution just as before. Any other kind of solution will be called *nontrivial.* The subject of differential equations is devoted to finding nontrivial solutions to differential equations, which is exactly what we will be doing here.

2.1.1. *The Characteristic Polynomial.*

How do we find solutions to (2.1)? Let's look at the equation and try to interpret what it means. Suppose that y is a solution to (2.1). This equation is a *linear combination* of y, y', and y'' (a linear combination of two functions f and g is a sum of the form: $c_1 f + c_2 g$ where c_1 and c_2 are any two numbers), so that probably means that taking derivatives of y shouldn't change y other than maybe multiplying it by a constant. What function do you know of that does that? Hopefully you were thinking of the function $y = e^{rx}$ since $y' = re^{rx}$ and $y'' = r^2 e^{rx}$. Suppose that $y = e^{rx}$ is indeed a solution to (2.1). Then taking derivative we get:

$$y' = re^{rx}$$

and

$$y'' = r^2 e^{rx}.$$

Plugging this in we have

$$ay'' + by' + cy = ar^2 e^{rx} + bre^{rx} + ce^{rx} = (ar^2 + br + c)e^{rx} = 0$$

which implies that, since e^{rx} is never zero:

$$ar^2 + br + c = 0. \tag{2.2}$$

Thus we have at least one solution of the form e^{rx} and usually two. Notice here that r may be a complex number. There are three cases:

(1) 2 distinct real roots
(2) a repeated real root
(3) complex conjugates

In the next three subsections we will cover these three cases. Equation (2.2) is called the **characteristic polynomial** of (2.1).

2.1.2. *Distinct Real Roots.*

Suppose that we are in the case in which the characteristic polynomial gives us two distinct real roots. Let's call them r_1 and r_2. Then we have two solutions: one of the form $e^{r_1 x}$ and $e^{r_2 x}$.

Theorem 1 (Distinct Real Roots)**.** *Suppose that the characteristic polynomial of $ay'' + by' + cy = 0$ has two real distinct roots: r_1 and r_2. Then the general solution to $ay'' + by' + cy = 0$ is*

$$y_G = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

Proof. Since r_1 and r_2 are roots of the polynomial $ar^2 + br + c = 0$, we instantly know that:

$$ar_1^2 + br_1 + c = 0$$

and

$$ar_2^2 + br_2 + c = 0.$$

Knowing this, let's plug y_G into the differential equation. First calculate y_G' and y_G'':

$$y_G' = c_1 r_1 e^{r_1 x} + c_2 r_2 e^{r_2 x}$$

and

$$y_G'' = c_1 r_1^2 e^{r_1 x} + c_2 r_2^2 e^{r_2 x},$$

and plug them into the differential equation to get:

$$\begin{aligned} ay_G'' + by_G' + cy_G &= a(c_1 r_1^2 e^{r_1 x} + c_2 r_2^2 e^{r_2 x}) + b(c_1 r_1 e^{r_1 x} + c_2 r_2 e^{r_2 x}) + c(c_1 e^{r_1 x} + c_2 e^{r_2 x}) \\ &= (ar_1^2 + br_1 + c)c_1 e^{r_1 x} + (ar_2^2 + br_2 + c)c_2 e^{r_2 x} \\ &= 0(c_1 e^{r_1 x}) + 0(c_2 e^{r_2 x}) = 0 \end{aligned}$$

Thus y_G is the general solution to our differential equation.

□

I used the notation y_G here to denote the general solution. I will be using this notation from here on out to mean general solution.

Example 1. *Solve the differential equation*

$$y'' - y' - 2y = 0.$$

Solution. *Start by finding the characteristic polynomial:*

$$r^2 - r - 2 = 0.$$

Now this factors into:

$$r^2 - r - 2 = (r-2)(r+1) = 0$$

so our roots are: $r = -1, 2$. *Thus our general solution is of the form:*

$$y_G = c_1 e^{-x} + c_2 e^{2x}.$$

◇

Remark. *It may seem kind of akward when one of the roots of the characteristic polynomial is* $r = 0$, *because then you get a solution of the form:*

$$y_G = c_1 e^{r_1 x} + c_2 e^{0x} = c_1 e^{r_1 x} + c_2.$$

This is, in fact, the correct solution and is not akward at all as $y_1 = e^{r_1 x}$ *and* $y_2 = 1$ *are* **linearly independent functions** *(we will define this term in the next section).*

2.1.3. *Repeated Real Roots.*

Now that we have covered when we have two distinct real roots, we need to handle the case when there is a repeated real root. So suppose that our characteristic polynomial has only one root: k (i.e. the polynomial factors as $(r-k)^2 = 0$). I claim that the general solution is

$$y_G = c_1 e^{kx} + c_2 x e^{kx}.$$

To prove this, we will have to use a method similar to variation of parameters for first order differential equations. The method we are going to use is actually called *reduction of order* which we will study in section 2.4. First note that if the characteristic polynomial has a repeated root k, it can be written in the form:

$$(r-k)^2 = r^2 - 2kr + k^2 = 0$$

and thus the differential equation looks like:

$$y'' - 2ky' + k^2 y = 0$$

Theorem 2 (Repeated Real Roots)**.** *Suppose that the characteristic polynomial of* $y'' - 2ky' + k^2 y = 0$ *has one repeated root:* k. *Then the general solution to* $y'' - 2ky' + k^2 y = 0$ *is*

$$y_G = c_1 e^{kx} + c_2 x e^{kx}.$$

Proof. We already know that $y_1 = e^{kx}$ is a solution to $y'' - 2ky' + k^2y = 0$, but we should be able to find a second linearly independent solution to it. To find the general solution, assume that it has the form $y_G = uy_1 = ue^{kx}$. Then

$$y'_G = u'e^{kx} + kue^{kx}$$

and

$$y''_G = u''e^{kx} + 2ku'e^{kx} + k^2ue^{kx}.$$

Now plug this in, and we get:

$$\begin{aligned} y''_G - 2ky'_G + k^2y_G &= u''e^{kx} + 2ku'e^{kx} + k^2ue^{kx} - 2k(u'e^{kx} + kue^{kx}) + k^2ue^{kx} \\ &= u''e^{kx} + 2ku'e^{kx} + k^2uekx - 2ku'e^{kx} - 2k^2ue^{kx} \\ &= u''e^{kx} = 0 \end{aligned}$$

Thus we are left with

$$u'' = 0$$

which gives

$$u = (c_1 + c_2x).$$

Plugging this into the formula for our general solution we get:

$$y_G = uy_1 = (c_1 + c_2x)e^{kx} = c_1e^{kx} + c_2xe^{kx}$$

as predicted. □

Example 2. *Solve the differential equation*

$$2y'' + 4y' + 2y = 0.$$

Solution. *The characteristic polynomial for this differential equation is:*

$$2r^2 + 4r + 2 = 0.$$

Dividing both sides by 2 *and factoring we have*

$$r^2 + 2r + 1 = (r + 1)^2 = 0$$

so the root is $r = 1$. *Thus our general solution is:*

$$y_G = c_1e^{-x} + c_2xe^{-x}.$$

◇

2.1.4. *Complex Roots.*

Lastly, we come to the case in which the roots of the characteristic polynomial are a conjugate pair of complex numbers, say $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$. This means that our two solutions should look like:

$$y_1 = e^{(\alpha+i\beta)x}$$

and

$$y_2 = e^{(\alpha-i\beta)x}.$$

This is correct, but not really useful as it only gives us data in the complex plane, not strictly real data (the output can be complex, not just real). However, there is a way of maniplating these two solutions, by taking complex-linear combinations of them, to get solutions that only give real data. First let's rewrite y_1 and y_2 using *Euler's formula* ($e^{ix} = \cos x + i \sin x$).

$$y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$$

and

$$y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x}e^{-i\beta x} = e^{\alpha x}(\cos\beta x + i\sin -\beta x) = e^{\alpha x}(\cos\beta x - i\sin\beta x).$$

Now consider the following:

$$y_3 = \frac{1}{2}(y_1 + y_2) = \frac{1}{2}(e^{\alpha x}(\cos\beta x + i\sin\beta x) + e^{\alpha x}(\cos\beta x - i\sin\beta x)) = e^{\alpha x}\frac{1}{2}(2\cos\beta x) = e^{\alpha x}\cos\beta x$$

and

$$y_4 = \frac{1}{2i}(y_1 - y_2) = \frac{1}{2}(e^{\alpha x}(\cos\beta x + i\sin\beta x) - e^{\alpha x}(\cos\beta x - i\sin\beta x)) = e^{\alpha x}\frac{1}{2}(2i\sin\beta x) = e^{\alpha x}\sin\beta x.$$

With these two new solutions (they are still solutions since they are just linear combinations of y_1 and y_2) we can construct a new general solution that only outputs real data:

$$y_G = c_1e^{\alpha x}\cos\beta x + c_2e^{\alpha x}\sin\beta x.$$

Let's verify that this is still a solution of the differential equation:

If the characteristic polynomial indeed has the roots $\alpha \pm i\beta$, then it is of the form $r^2 - 2\alpha r + (\alpha^2 + \beta^2) = 0$, and thus the differential equation looks like: $y'' - 2\alpha y' + (\alpha^2 + \beta^2)y = 0$. Now for the derivatives:

$$y_G = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x),$$

$$y_G' = e^{\alpha x}(\alpha c_1 \cos \beta x + \beta c_2 \cos \beta x + \alpha c_2 \sin \beta x - \beta c_1 \sin \beta x),$$

and

$$y_G'' = e^{\alpha x}(\alpha^2 c_1 \cos \beta x - \beta^2 c_1 \cos \beta x + \alpha^2 c_2 \sin \beta x - \beta^2 c_2 \sin \beta x + 2\alpha\beta c_2 \cos \beta x - 2\alpha\beta c_1 \sin \beta x).$$

Plugging these in we get:

$$\begin{aligned}
y_G'' - 2\alpha y_G' + (\alpha^2 + \beta^2) y_G &= e^{\alpha x}(\alpha^2 c_1 \cos \beta x - \beta^2 c_1 \cos \beta x + \alpha^2 c_2 \sin \beta x - \beta^2 c_2 \sin \beta x + 2\alpha\beta c_2 \cos \beta x - 2\alpha\beta c_1 \sin \beta x) \\
&\quad -2\alpha(e^{\alpha x}(\alpha c_1 \cos \beta x + \beta c_2 \cos \beta x + \alpha c_2 \sin \beta x - \beta c_1 \sin \beta x)) \\
&\quad +(\alpha^2 + \beta^2)(e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)) \\
&= e^{\alpha x}\left[(\alpha^2 c_1 - \beta^2 c_1 + 2\alpha\beta c_2 - 2\alpha^2 c_1 - 2\alpha\beta c_2 + \alpha^2 c_1 + \beta^2 c_1) \cos \beta x\right. \\
&\quad \left. +(\alpha^2 c_2 - \beta^2 c_2 - 2\alpha\beta c_1 - 2\alpha^2 c_2 + 2\alpha\beta c_1 + \alpha^2 c_2 + \beta^2 c_2) \sin \beta x\right] \\
&= e^{\alpha x}\left[0 \cos \beta x + 0 \sin \beta x\right] = 0
\end{aligned}$$

Thus y_G is our general solution.

Theorem 3 (Complex Conjugate Roots)**.** *Suppose that the characteristic polynomial of $ay'' + by' + cy = 0$ has two real complex roots: $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$. Then the general solution to $ay'' + by' + cy = 0$ is*

$$y_G = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x.$$

Proof. The proof is in the discussion above. □

Remark. *Technically we have not completely proven that the above are the general solutions to their respective equations because we have not shown that the two solutions in each general solution form a linearly independent set of functions. This will be done in the next section.*

Example 3. *Solve the differentiale equation*

$$y'' + 9y = 0.$$

Solution. *The characteristic polynomial is*

$$r^2 + 9 = 0,$$

thus the roots are $r = \pm 3i$. So our general solution is:

$$y_G = c_1 \cos 3x + c_2 \sin 3x.$$

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Exercises.

Solve the following differential equations:

(1) $y'' - y' - 6y = 0$

(2) $y'' + 2y' = 0$

(3) $y'' + 2y' + y = 0$

(4) $y'' + 9y = 0$

(5) $y'' - 6y' + 13y = 0$

(6) $y'' - 4y' + 5y = 0$

(7) $y'' - 4y' + 4y = 0$

(8) $y'' + 6y' + 10y = 0$

(9) $y'' + y' = 0$

(10) $y'' + 6y' + 13y = 0$

(11) $10y'' - 3y' - y = 0$

(12) $4y'' + 4y' + 10y = 0$

Solve the initial value problems:

(13) $y'' - 4y' + 3y = 0; y(0) = -1, y'(0) = 3$

(14) $y'' + 4y = 0; y(\pi) = 1, y'(\pi) = -4$

(15) $6y'' - y' - y = 0; y(0) = 10, y'(0) = 0$

(16) $4y'' - 4y' - 3y = 0; y(0) = \frac{13}{12}, y'(0) = \frac{23}{24}$

(17) $y'' + 7y' + 12y = 0; y(0) = -1, y'(0) = 0$

(18) $36y'' - 12y' + y = 0; y(0) = 3, y'(0) = \frac{5}{2}$

Challenge Problems:

(19)

(a) Suppsoe that y is a solution of the differential equation

$$ay'' + by' + cy = 0.$$

Let $z(x) = y(x - x_0)$ where x_0 is an arbitrary real number. Show that

$$az'' + bz' + cz = 0.$$

(b) Let $z_1(x) = y_1(x - x_0)$ and $z_2(x) = y_2(x - x_0)$, where $y_G = c_1y_1 + c_2y_2$ is the general solution to $ay'' + by' + cy = 0$. Show that $z_G = c_1z_1 + c_2z_2$ is also a general solution of $ay'' + by' + cy = 0$.

(20) Prove that if the characteristic equation of

$$ay'' + by' + cy = 0$$

has a repeated negative root or two roots with negative real parts, then every solution of $ay'' + by' + cy = 0$ approaches zero as $x \to \infty$.

(21) Consider the differential equation $ay'' + by' + cy = d$ where d is any real number. Find it's general solution. *Hint:* First try to find a *particular solution* by finding a function to plug in that gives d as an output, then use the principle of superposition.

(22) Consider the differential equation $ay'' + by' + cy = 0$ with $a > 0$. Find conditions on a, b, and c such that the roots of the characteristic polynomial are:

(a) real, different, and negative.

(b) real with opposite signs.

(c) real, different, and positive.

(23) Let f and g be any twice differentiable functions. Suppose that $L[y] = p(x)y'' + q(x)y' + r(x)y$. Show that $L[f + g] = L[f] + L[g]$.

2.2. The Wronskian.

2.2.1. *General Homogeneous Linear Equations.*

In this section we will study the general solutions of the general homogeneous linear differential equation:

$$y'' + q(x)y' + r(x)y = 0. \tag{2.3}$$

First we need to make clear a few definitions:

Definition 1 (Linear Combination)**.** *Suppose that y_1 and y_2 are two functions. Then a linear combination of y_1 and y_2 is a function of the form*

$$c_1y_1 + c_2y_2$$

where c_1 and c_2 are arbitrary constants.

Definition 2 (Fundamental Set of Solutions)**.** *Suppose that y_1 and y_2 are two solutions of* (2.3)*. The set $\{y_1, y_2\}$ is called a fundamental set of solutions for* (2.3) *if every solution of* (2.3) *can be written as a linear combination of y_1 and y_2.*

Definition 3 (General Solution)**.** *Suppose that $\{y_1, y_2\}$ is a fundamental set of solutions to* (2.3)*. Then the general solution to* (2.3) *is*

$$y_G = c_1y_1 + c_2y_2.$$

This definition of a fundamental set of solutions really helps nail down what a general solution to (2.3) is, however how can we tell if two solutions of (2.3) form a fundamental set of solution? The answer is: "if they are *linearly independent.*"

2.2.2. *The Wronskian.*

Definition 4 (Linearly Independent)**.** *Two functions y_1 and y_2 are said to be linearly independent on the interval (a, b) if neither is a constant multiple of the other on (a, b), i.e. for any value of c the following equation never holds for all $x \in (a, b)$: $y_1(x) = cy_2(x)$. If two functions are not linearly independent, they are called linearly dependent.*

Remark. *Another way to show linear dependence is to show that*

$$c_1y_1(x) + c_2y_2(x) = 0$$

for all $x \in (a, b)$ where c_1 and c_2 are not both zero.

Ok, this helps us, since if two solutions, y_1 and y_2, are not linearly independent, the following holds (suppose that $y_1 = ky_2$):

$$c_1y_1 + c_2y_2 = c_1(ky_2) + c_2y_2 = (c_1k + c_2)y_2,$$

which is no longer a linear combination of two solutions, which is what we should have since the fact that we have two derivatives tells us that there should be two linearly independent solutions (think of each derivative as a degree of freedom). We have made some progress, however, it might not be easy to show that two functions are linearly independent (or even linearly dependent). We would like a way that makes it easy to check whether or not functions are linearly independent. This leads us to the Wronskian, named after the the Polish mathematician Wronski:

Definition 5 (The Wronskian)**.** *The Wronskian of two functions y_1 and y_2 at the point x is given by:*

$$W(x; y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

When it is clear which functions are involved, we will often shorten the notation to $W(x)$.

To see how this is useful to us, consider the following theorem:

Theorem 1. *Let y_1 and y_2 be functions on an interval (a, b). If the functions are linearly dependent on (a, b), then $W(x; y_1, y_2) = 0$ for all $x \in (a, b)$. Thus if $W(x; y_1, y_2) \neq 0$ for at least one point in (a, b), the functions are linearly independent.*

Notice that the second statement is just the *contrapositive* of the first, so we only have to prove the first statement (which is MUCH easier to prove).

Proof. Suppose that the functions y_1 and y_2 are linearly dependent on an interval (a, b). Then for all $x \in (a, b)$ we have

$$y_1(x) = cy_2(x),$$

thus the Wronskian is:

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} cy_2(x) & y_2(x) \\ cy_2'(x) & y_2'(x) \end{vmatrix} = cy_2(x)y_2'(x) - cy_2'(x)y_2(x) \equiv 0,$$

as claimed. ∎

However, do not assume that this theorem means that if the Wronskian of two functions is identically zero that the two functions are linearly dependent (this is usually said as "the converse is not true"). Here is a counter example to the converse:

Example 1. *Let* $y_1(x) = x^2$ *and* $y_2(x) = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$. *Show that* y_1 *and* y_2 *are linearly independent on* $\mathbb{R}$.

Solution. *The easiest way to check linear independence is to show that their Wronskian is nonzero for at least one point in* $\mathbb{R}$. *First note that* $y_2'(0)$ *exists and is zero. When* $x \geq 0$ *we have:*

$$W(x) = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} \equiv 0,$$

and for $x < 0$ *we have:*

$$W(x) = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} \equiv 0.$$

So the "Wronskian test" for linear independence fails because $W(x) \equiv 0$ *on* $\mathbb{R}$. *So, suppose that* y_1 *and* y_2 *are linearly dependent on* $\mathbb{R}$, *i.e.*

$$c_1 y_1(x) + c_2 y_2(x) = c_1 x^2 + c_2 x|x| = 0$$

with c_1 *and* c_2 *are not both zero. Plugging in* $x = 1$ *we get the equation:* $c_1 + c_2 = 0$*; and plugging in* $x = -1$ *we get the equation* $c_1 - c_2 = 0$. *So we have the system of equations:*

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 - c_2 = 0 \end{cases}$$

which has the solution $c_1 = c_2 = 0$, *a contradiction to our assumption. Thus* y_1 *and* y_2 *are linearly independent on* $\mathbb{R}$.

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Now, we will state a "unifying theorem" without proof:

Theorem 2. *Suppose that* q *and* r *are continuous on the interval* (a, b), *and let* y_1 *and* y_2 *be solutions of* (2.3) *on* (a, b). *Then the following are equivalent (either all the statements are true, or they are all false):*

(a) *The general solution of* (2.3) *is* $y_G = c_1 y_1 + c_2 y_2$.
(b) $\{y_1, y_2\}$ *is a fundamental set of solutions of* (2.3) *on* (a, b).
(c) $\{y_1, y_2\}$ *is linearly independent on* (a, b).
(d) *The Wronskian of* $\{y_1, y_2\}$ *is nonzero at some point in* (a, b).
(e) *The Wronskian of* $\{y_1, y_2\}$ *is nonzero at all points in* (a, b).

This is a truly remarkable theorem as it allows us to prove so much with so little (the equivalence of (*d*) and (*e*) for example)!!!

In the previous section we found what the general solutions of constant coefficient second order linear differential equations are, except that we didn't completely prove that the two solutions were linearly independent. Let's do that now.

Theorem 3. *Consider the differential equation*

$$ay'' + by' + cy = 0.$$

(a) *If the characteristic polynomial has two distinct real roots,* r_1 *and* r_2, *then* $y_1 = e^{r_1 x}$ *and* $y_2 = e^{r_2 x}$ *form a fundamental set of solutions for* $ay'' + by' + cy = 0$ *on* $\mathbb{R}$.
(b) *If the characteristic polynomial has a repeated real root,* k, *then* $y_1 = e^{kx}$ *and* $y_2 = xe^{kx}$ *form a fundamental set of solutions for* $ay'' + by' + cy = 0$ *on* $\mathbb{R}$.
(c) *If the characteristic polynomial has two complex conjugate roots,* $\alpha + i\beta$ *and* $\alpha - i\beta$, *then* $y_1 = e^{\alpha x}\cos\beta x$ *and* $y_2 = e^{\alpha x}\sin\beta x$ *form a fundamental set of solutions for* $ay'' + by' + cy = 0$ *on* $\mathbb{R}$.

Proof.

(a) Construct the Wronskian for y_1 and y_2:

$$W(x) = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix} = r_2 e^{r_1 x} e^{r_2 x} - r_1 e^{r_1 x} e^{r_2 x} = (r_2 - r_1) e^{r_1 x} e^{r_2 x} \neq 0 \qquad \forall x \in \mathbb{R},$$

since $r_1 \neq r_2$ and $e^{r_1 x}$ and $e^{r_2 x}$ are never zero.

(b) Construct the Wronskian for y_1 and y_2:

$$W(x) = \begin{vmatrix} e^{kx} & xe^{kx} \\ ke^{kx} & e^{kx} + kxe^{kx} \end{vmatrix} = e^{2kx} + kxe^{2kx} - kxe^{kx} = e^{2kx} \neq 0, \qquad \forall x \in \mathbb{R}.$$

(c) Left as an exercise to the reader.

□

Exercises.

Show that the following pairs of functions are linearly independent on the given interval:

(1) $y_1 = e^{ax}, y_2 = e^{bx}$, $a \neq b$, $(-\infty, \infty)$

(2) $y_1 = \cos ax, y_2 = \sin ax$, $(-\infty, \infty)$

(3) $y_1 = 1, y_2 = e^{ax}$, $a \neq 0$, $(-\infty, \infty)$

(4) $y_1 = e^{ax}\cos bx, y_2 = e^{ax}\sin bx$, $(-\infty, \infty)$

(5) $y_1 = \dfrac{1}{x}, y_2 = \dfrac{1}{x^3}$, $(0, \infty)$

Determine whether or not the following pair of functions is linearly independent or linearly dependent on the given interval:

(6) $y_1 = x^m, y_2 = |x|^m$, m a positive integer, $(-\infty, \infty)$

Challenge Problems:

(7) Show that if y_1 and y_2 are C^2 functions on the interval (a, b) and $W(x; y_1, y_2)$ has no zeros in (a, b), then the equation

$$\frac{1}{W(x; y_1, y_2)} \begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{vmatrix} = 0$$

can be written as

$$y'' + q(x)y' + r(x)y = 0$$

with $\{y_1, y_2\}$ as a fundamental set of solutions on (a, b). *Hint:* Expand the determinant by cofactors of its first column.

(8) Use Exercise 6 to find a differential equation whose fundamental set of solutions is the given pair of functions:

(a) $\{x, x \ln x\}$

(b) $\{x, e^{2x}\}$

(c) $\{\cosh x, \sinh x\}$

(d) $\{x^2 - 1, x^2 + 1\}$

(9) Prove the following:

Theorem 4 (Abel's Formula)**.** *Suppose that q and r are continuous on (a, b), let y_1 and y_2 be solutions of*

$$y'' + q(x)y' + r(x)y = 0$$

on (a, b). Let x_0 be any point in (a, b). Then

$$W(x; y_1, y_2) = W(x_0; y_1, y_2)e^{-\int_{x_0}^{x} q(t)\, dt}, \quad x \in (a, b).$$

Hint: Take the derivative of $W(x; y_1, y_2)$ and use the equations $y_1'' = -qy_1' - ry_1$ and $y_2'' = -qy_2' - ry_2$ to transform the derivative of W into a separable equation in terms of W. Then use the method on page 21 to deal with the initial value $W(x_0)$.

2.3. Non-Homogeneous Linear Equations.

In this section we will introduce non-homogeneous linear equations and study certain kinds. Recall that a non-homogeneous second order linear differential equation has the form:

$$y'' + q(x)y' + r(x)y = f(x) \tag{2.4}$$

where $f(x) \not\equiv 0$.

In previous sections we have been studying the same equation, but with $f(x) \equiv 0$. So how do we deal with this case? We need to somehow plug a function into the differential equation's left hand side $(y'' + q(x)y' + r(x)y)$ and produce $f(x)$. Let's suppose such a function exists and call it y_p for *particular solution*. Is this the general solution to (2.4)? The answer is no. The general solution is of the form $y_G = y_H + y_p$, where y_H is the general solution of the associated homogeneous equation:

$$y'' + q(x)y' + r(x)y = 0.$$

Why is this true? It is true because $0 + f(x) = f(x)$. Sound confusing? Observe the following:

We know that $y_H'' + q(x)y_H' + r(x)y_H = 0$ and that $y_p'' + q(x)y_p' + r(x)y_p = f(x)$, so let's assume that $y_G = y_H + y_p$ and plug y_G into (2.4):

$$y_G' = y_H' + y_p'$$

and

$$y_G'' = y_H'' + y_p'',$$

so

$$y_G''+q(x)y_G'+r(x)y_G = y_H''+y_p''+q(x)(y_H'+y_p')+r(x)(y_H+y_p) = (y_H''+q(x)y_H'+r(x)y_H)+(y_p''+q(x)y_p'+r(x)y_p) = 0+f(x) = f(x).$$

See? Easy as $0 + f(x) = f(x)$!!!

Example 1. *Show that $y_p = e^{2x}$ is a particular solution to $y'' + 2y' - 3y = 5e^{2x}$ and find the solution to the IVP*

$$y'' + 2y' - 3y = 5e^{2x}; y(0) = 5, y'(0) = 2.$$

Solution. *First let's check that $y_p = e^{2x}$ is the particular solution:*

$$y_p'' + 2y_p' - 3y_p = 4e^{2x} + 4e^{2x} - 3e^{2x} = 5e^{2x}.$$

Thus y_p is in fact the particular solution. Now to solve the IVP we need to first find the general solution to the differential equation. The characteristic polynomial for the associated homogeneous equation is

$$r^2 + 2r - 3 = (r + 3)(r - 1) = 0$$

so the homogeneous solution is $y_H = c_1e^{-3x} + c_2e^x$. Thus the general solution to the non-homogeneous equation is

$$y_G = c_1e^{-3x} + c_2e^x + e^{2x}.$$

To use the initial value we first need to know what y_G' is:

$$y_G' = -3c_1e^{-3x} + c_2e^x + 2e^{2x}.$$

Now plug in the initial values:

$$\begin{cases} y_G(0) = c_1 + c_2 + 1 = 5 \\ y_G'(0) = -3c_1 + c_2 + 2 = 2 \end{cases}$$

which simplifies to:

$$\begin{cases} y_G(0) = c_1 + c_2 = 4 \\ y_G'(0) = -3c_1 + c_2 = 0 \end{cases}$$

which yields the solution $c_1 = 1$, and $c_2 = 3$. So the solution to the IVP is

$$y = e^{-3x} + 3e^x + e^{2x}.$$

$\diamond$

2.3.1. *Superposition.*

Theorem 1 (Principle of Superposition)**.** *Suppose that y_{p_1} is a particular solution of*

$$y'' + q(x)y' + r(x)y = f_1(x)$$

and that y_{p_2} is a particular solution of

$$y'' + q(x)y' + r(x)y = f_2(x).$$

Then

$$y_p = y_{p_1} + y_{p_2}$$

is a particular solution of

$$y'' + q(x)y' + r(x)y = f_1(x) + f_2(x).$$

This can be generalized quite easily to the equation

$$y'' + q(x)y' + r(x)y = f(x)$$

where $f(x) = f_1(x) + \cdots + f_n(x)$.

Example 2. *Given that $y_{p_1} = \frac{1}{15}x^4$ is a particular solution of*

$$x^2y'' + 4xy' + 2y = 2x^4$$

and that $y_{p_2} = \frac{1}{3}x^2$ is a particular solution of

$$x^2y'' + 4xy' + 2y = 4x^2,$$

find a particular solution to

$$x^2y'' + 4xy' + 2y = 2x^4 + 4x^2.$$

Solution. *By the principle of superposition, our particular solution should be*

$$y_p = y_{p_1} + y_{p_2} = \frac{1}{15}x^4 + \frac{1}{3}x^2.$$

Let's verify this by plugging it into the differential equation:

$$\begin{aligned}
x^2y_p'' + 4xy_p' + 2y_p &= x^2\left(\frac{12}{15}x^2 + \frac{2}{3}\right) + 4x\left(\frac{4}{15}x^3 + \frac{2}{3}x\right) + 2\left(\frac{1}{15}x^4 + \frac{1}{3}x^2\right) \\
&= \frac{12}{15}x^4 + \frac{2}{3}x^2 + \frac{16}{15}x^4 + \frac{8}{3}x^2 + \frac{2}{15}x^4 + \frac{2}{3}x^2 \\
&= \frac{30}{15}x^4 + \frac{12}{3}x^2 \\
&= 2x^4 + 4x^2
\end{aligned}$$

as required.

◇

2.3.2. *Introduction to Undetermined Coefficients.*

In this section we will begin investigating a technique for solving equations of the form

$$ay'' + by' + cy = P_n(x)$$

where a, b, and c are constants, $a \neq 0$, and $P_n(x)$ is a polynomial of degree n.

Theorem 2 (Method of Undetermined Coefficients)**.** *Suppose that we have a differential equation of the form*

$$ay'' + by' + cy = P_n(x), \quad a \neq 0$$

where $P_n(x)$ is a polynomial of degree n. Then the particular solution to the differential equation is of the form:

If $c \neq 0$)

$$y_p = A_nx^n + A_{n-1}x^{n-1} + \cdots + A_2x^2 + A_1x + A_0$$

where the A_i, $i = 0, ..., n$ are constants to be determined by plugging y_p into the differential equation and equating coefficients.

If $c = 0$ & $b \neq 0$)

$$y_p = A_nx^{n+1} + A_{n-1}x^n + \cdots + A_1x^2 + A_0x$$

where the A_i, $i = 0, ..., n$ are constants to be determined by plugging y_p into the differential equation and equating coefficients.

If $b = c = 0$) *Just integrate both sides of the differential equation twice.*

Example 3. *Find the general solution of the given equations:*

(a) $y'' - 4y' + 5y = 5x + 1$
(b) $y'' + y' = x^2 + 2x + 1$

Solution.

(a) *The characteristic polynomial for the corresponding homogeneous equation is*

$$r^2 - 4r + 5 = 0$$

which has roots:

$$r = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm i$$

so

$$y_H = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x.$$

Now by the theorem above assume that $y_p = A_1 x + A_0$*. Then plugging in gives:*

$$y_p'' - 4y_p' + 5y_p = 0 - 4(A_1) + 5(A_1 x + A_0) = (5A_1)x + (5A_0 - 4A_1) = 5x + 1$$

and comparing coefficients gives the system:

$$\begin{cases} & & 5A_1 & = & 5 \\ 5A_0 & - & 4A_1 & = & 1 \end{cases}$$

which yields the solution $A_0 = 1$ *and* $A_1 = 1$*. Thus the particular solution is*

$$y_p = x + 1$$

and the general solution is

$$y_G = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x + x + 1.$$

(b) *The characteristic polynomial for the corresponding homogeneous equation is*

$$r^2 + r = r(r + 1) = 0$$

which has roots:

$$r = -1, 0$$

so

$$y_H = c_1 e^{-x} + c_2.$$

Now since $c = 0$ *and* $b \neq 0$ *in this equation, we guess that* $y_p = A_2 x^3 + A_1 x^2 + A_0 x$*. Then plugging in gives:*

$$y_p'' + y_p' = 6A_2 x + 2A_1 + 3A_2 x^2 + 2A_1 x + A_0 = (3A_2)x^2 + (6A_2 + 2A_1)x + (2A_1 + A_0) = x^2 + 2x + 1$$

and comparing coefficients gives the system:

$$\begin{cases} & & & & 3A_2 & = & 1 \\ & & 2A_1 & + & 6A_2 & = & 5 \\ A_0 & + & 2A_1 & & & = & 1 \end{cases}$$

which yields the solution $A_0 = -2$, $A_1 = \frac{3}{2}$, *and* $A_2 = \frac{1}{3}$*. Thus the particular solution is*

$$y_p = \frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x$$

and the general solution is

$$y_G = c_1 e^{-x} + c_2 + \frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x.$$

◇

Exercises. Find the general solution, and if an initial value is given, solve the IVP:

(1) $y'' + 5y' - 6y = -18x^2 + 18x + 22$

(2) $y'' + 8y' + 7y = 7x^3 + 24x^2 - x - 8$

(3) $y'' + 2y' + 10y = 10x^3 + 6x^2 + 26x + 4; y(0) = 2, y'(0) = 9$

(4) $y'' + 6y' + 10y = 20x + 22; y(0) = 2, y'(0) = -2$

(5) $y'' - y' - 6y = 2$

(6) $y'' + 3y' + 2y = 4x^2$

(7) $y'' + 2y = -4$

(8) $y'' + y = 3x^2$

(9) Verify that $y_p = \sin 2x$ is a solution of $y'' - y = -5\sin 2x$ and use it to find the general solution.

Challenge Problems:

(10) Prove Theorem 1 of this section.

(11) Generalize Theorem 1 to the case where $f(x) = f_1(x) + \cdots + f_n(x)$. (You don't have to prove the generalization, just state it.)

(12) Show that the equation

$$b_0x^2y'' + b_1xy' + b_2y = cx^a$$

has a solution of the form $y = Ax^a$ provided that

$$b_0a(a-1) + b_1a + b_2 \neq 0.$$

(13) If $c \neq 0$ and d is a constant, show that a solution to the equation

$$ay'' + by' + cy = d$$

is $y = \dfrac{d}{c}$. What would the solution look like if $c = 0$?

2.4. Reduction of Order.

The method of reduction of order is a way to solve a non-homogeneous second order linear differential equation:

$$p(x)y'' + q(x)y' + r(x)y = f(x),$$

provided that we know a nontrivial solution to the associated homogeneous equation:

$$p(x)y'' + q(x)y' + r(x)y = 0.$$

Reduction of order does exactly what it sounds like it does: it reduces the order of the differential equation. More precisely, it reduces it from a second order equation to a first order equation. Let's see how this method gets its name:

Suppose that y_1 is a nontrivial solution to $p(x)y'' + q(x)y' + r(x)y = 0$, and let $y = uy_1$ where u is a function to be determined. Plug y into $p(x)y'' + q(x)y' + r(x)y = f(x)$ ($y' = u'y_1 + uy_1'$, $y'' = u''y_1 + 2u'y_1' + uy_1''$):

$$p(x)(u''y_1+2u'y_1'+uy_1'')+q(x)(u'y_1+uy_1')+r(x)(uy_1) = (p(x)y_1)u''+(2p(x)y_1'+q(x)y_1)u'+(p(x)y_1''+q(x)y_1'+r(x)y_1)u = f(x).$$

But notice that the coefficient in front of u is zero since $p(x)y_1'' + q(x)y_1' + r(x)y_1 = 0$, thus our equation reduces to:

$$(p(x)y_1)u'' + (2p(x)y_1' + q(x)y_1)u' = f(x).$$

This may not look like a first order equation, but believe it or not, it is! To see this just let $v = u'$, then $v' = u''$ and:

$$(p(x)y_1)v' + (2p(x)y_1' + q(x)y_1)v = f(x),$$

which is a first order linear differential equation in v. So by solving for v, we get u' which we can integrate to find u, and thus y.

Remark. *It is not necessary to make the substitution $v = u'$, I did it merely to illustrate a point. However, it is not incorrect to do it either. I will use the substitution in this section to make the examples look simpler.*

Example 1. *Find the general solution of*

$$t^2y'' - 4ty' + 6y = t^7, t > 0$$

given that $y_1 = t^2$.

Solution. *Set $y = uy_1 = ut^2$. Then $y' = u't^2 + 2ut$ and $y'' = u''t^2 + 4u't + 2u$. Plug these into the equation to get:*

$$t^2(u''t^2 + 4u't + 2u) - 4t(u't^2 + 2ut) + 6(ut^2) = t^4u'' = t^7,$$

which gives:

$$u'' = t^3.$$

There is no need to make the change of variable $v = u'$ here, so:

$$u' = \int t^3\,dt = \frac{1}{4}t^4 + c_2$$

and

$$u = \int \frac{1}{4}t^4 + c_2\,dt = \frac{1}{20}t^5 + c_2t + c_1.$$

Thus our general solution is:

$$y_G = uy_1 = \left(\frac{1}{20}t^5 + c_2t + c_1\right)t^2 = c_1t^2 + c_2t^3 + \frac{1}{20}t^7.$$

◇

If the differential equation you are given is second order homogeneous, and you only have one of the homogeneous solutions, you can actually use the method of reduction of order to find the other homogeneous solution! This is a particularly useful fact, especially when the equation you have is difficult. (This is actually how we found the second solution to the constant coefficient homogeneous equation when we have repeated roots!) Here is an example:

Example 2. *Find the general solution to*

$$xy'' - (4x + 1)y' + (4x + 2)y = 0, x > 0$$

given that $y_1 = e^{2x}$ is a solution.

Solution. *As usual let* $y = uy_1 = ue^{2x}$*. Then* $y' = u'e^{2x} + 2ue^{2x}$ *and* $y'' = u''e^{2x} + 4u'e^{2x} + 4ue^{2x}$*. Plug these in:* $x(u''e^{2x} + 4u'e^{2x} + 4ue^{2x}) - (4x+1)(u'e^{2x} + 2ue^{2x}) + (4x+2)ue^{2x} = xe^{2x}u'' + (4x - (4x+1))e^{2x}u' = xe^{2x}u'' - e^{2x}u' = 0$. *Let* $v = u'$ *and divide by* xe^{2x}*, then*

$$v' - \frac{1}{x}v = 0$$

which is separable into:

$$\frac{v'}{v} = \frac{1}{x}$$

thus integrating both sides gives:

$$\ln|v| = \ln x + c_2$$

or

$$v = c_2 x$$

which makes

$$u = \int v\,dx = c_2x^2 + c_1$$

so that

$$y_G = uy_1 = c_1e^{2x} + c_2x^2e^{2x}.$$

◇

Exercises. Find the general solution given one of the homogeneous solutions. If an initial value is given, solve the IVP.

(1) $x^2y'' + xy' - y = \frac{4}{x^2}$; $y_1 = x$

(2) $x^2y'' - xy' + y = x$; $y_1 = x$

(3) $y'' + 4xy' + (4x^2 + 2)y = 8e^{-x(x+2)}$; $y_1 = e^{-x^2}$

(4) $x^2y'' + 2x(x-1)y' + (x^2 - 2x + 2)y = x^3e^{2x}$; $y_1 = xe^{-x}$

(5) $x^2y'' + 2xy' - 2y = x^2; y(1) = \frac{5}{4}, y'(1) = \frac{3}{2}$; $y_1 = x$

(6) $(3x-1)y'' - (3x+2)y' - (6x-8)y = 0; y(0) = 2, y'(0) = 3$; $y_1 = e^{2x}$

(7) $t^2y'' + 2ty' - 2y = 0$; $y_1 = t$

(8) $4xy'' + 2y' + y = 0$; $y_1 = \sin\sqrt{x}$

(9) $(x^2 - 2x)y'' + (2 - x^2)y' + (2x - 2)y = 0$; $y_1 = e^x$

(10) $4x^2(\sin x)y'' - 4x(x\cos x + \sin x)y' + (2x\cos x + 3\sin x)y = 0$; $y_1 = x^{\frac{1}{2}}$

(11) $t^2y'' + 3ty' + y = 0$; $y_1 = t^{-1}$

(12) $(x-1)y'' - xy' + y = 0$; $y_1 = e^x$

Challenge Problems:

(13) Suppose that $q(x)$ and $r(x)$ are continuous on (a,b). Let y_1 be a solution of

$$y'' + q(x)y' + r(x)y = 0$$

that has no zeros on (a,b), and let x_0 be in (a,b). Use reduction of order to show that y_1 and

$$y_2 = y_1\int_{x_0}^{x} \frac{1}{y_1^2} e^{-\int_{x_0}^{t} q(s)\,ds} dt$$

form a fundamental set of solutions for $y'' + q(x)y' + r(x)y = 0$ on (a,b).

(14) Recall the Riccati equation

$$y' = p(x)y^2 + q(x)y + r(x).$$

Assume that q and r are continuous and p is differentiable.

(a) Show that y is a solution of the Riccati equation if and only if $y = -\dfrac{z'}{pz}$, where

$$z'' - \left[q(x) + \frac{p'(x)}{p(x)}\right] z' + p(x)r(x)z = 0.$$

(b) Show that the general solution of the Riccati equation is

$$y = -\frac{c_1 z_1' + c_2 z_2'}{p(c_1 z_1 + c_2 z_2)}$$

where $\{z_1, z_2\}$ is a fundamental set of solutions to the differential equation for z and c_1 and c_2 are arbitrary constants.

(15) The differential equation

$$xy'' - (x + N)y' + Ny = 0,$$

where N is a nonnegative integer, has an exponential solution and a polynomial solution.

(a) Verify that one solution is $y_1 = e^x$.

(b) Show that a second solution has the form $y_2 = ce^x \int x^N e^{-x}\, dx$. Calculate y_2 for $N = 1$ and $N = 2$; convince yourself that, with $c = -\dfrac{1}{N!}$,

$$y_2 = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^N}{N!}.$$

(If you have taken a calculus course in which Taylor series were covered, you might recognize this as the first $N + 1$ terms of the Maclauren series for e^x (that is, for y_1)!)

(16) The differential equation

$$y'' + \delta(xy' + y) = 0$$

arises in the study of the turbulent flow of a uniform stream past a circular sylinder. Verify that $y_1 = e^{-\frac{\delta x^2}{2}}$ is one solution, and then find the general solution in the form of an integral.

2.5. Cauchy-Euler Equations.

In this section we will study a special class of second order homogeneous linear differential equations called Cauchy-Euler equations.

Definition 1 (Cauchy-Euler Equation)**.** *A second order differential equation is said to be Cauchy-Euler if it is of the form*

$$ax^2y'' + bxy' + cy = 0, \tag{2.5}$$

where $a \neq 0$*.*

This equation actually has what it called a singular point at $x = 0$, which will be dealt with in Chapter 6. In order to avoid any complications the singular point may create, in this section, we restrict ourselves to $x > 0$.

2.5.1. *Indicial Equation.*

Just as in Section 2.1, there is a nice and easy way to solve Cauchy-Euler equations. Notice that with each derivative in the equation, you gain a power of x. Thus a solution to this equation must lose a power of x with each derivative. What function do you know of that does that? The answer is, of course, the simplest function, $y = x^m$! Observe:

$$y' = mx^{m-1}$$

and

$$y'' = m(m-1)x^{m-2}$$

so plugging them in we get:

$$ax^2(m(m-1)x^{m-2}) + bx(mx^{m-1}) + cx^m = am(m-1)x^m + bmx^m + cx^m = 0$$

and since $x > 0$ it is ok to divide by x^m, giving us:

$$am(m-1) + bm + c = 0. \tag{2.6}$$

Equation (2.6) is called the **indicial equation** for equation (2.5). Just as with any polynomial, this polynomial can have three types of roots:

(1) distinct real roots
(2) repeated real roots
(3) complex conjugate roots.

As in Section 2.1, let's examine what solutions we get with each case.

2.5.2. *Distinct Real Roots.*

Theorem 1 (Distinct Real Roots)**.** *Suppose that the indicial polynomial of* $ax^2y'' + bxy' + cy = 0$ *has two real distinct roots:* m_1 *and* m_2*. Then the general solution to* $ax^2y'' + bxy' + cy = 0$ *is*

$$y_G = c_1x^{m_1} + c_2x^{m_2}.$$

Example 1. *Solve the differential equation*

$$x^2y'' - 6xy' + 10y = 0.$$

Solution. *First let's begin by finding the indicial equation:*

$$m(m-1) - 6m + 10 = m^2 - 7m + 10 = (m-2)(m-5) = 0.$$

Since the roots of the indicial equation are 2 *and* 5*, the solution to the differential equation is:*

$$y_G = c_1x^2 + c_2x^5.$$

◇

2.5.3. *Repeated Real Roots.*

Theorem 2 (repeated Real Roots)**.** *Suppose that the indicial polynomial of $ax^2y'' + bxy' + cy = 0$ has one repeated root: k. Then the general solution to $ax^2y'' + bxy' + cy = 0$ is*

$$y_G = c_1x^k + c_2x^k \ln x.$$

Proof. First note that if the indicial equation has a repeated root, then it has the form:

$$m(m-1) - (2k-1)m + k^2 = 0,$$

and so the differential equation looks like:

$$x^2y'' - (2k-1)xy' + k^2y = 0.$$

Then this is just a routine application of reduction of order with $y_1 = x^k$. □

Example 2. *Find the general solution of*

$$x^2y'' + 5xy' + 4y = 0.$$

Solution. *The indicial equation is*

$$m(m-1) + 5m + 4 = m^2 + 4m + 4 = (m+2)^2 = 0.$$

So the repeated root of the equation is $m = -2$, and so the general solution is

$$y_G = c_1x^{-2} + c_2x^{-2} \ln x.$$

◇

2.5.4. *Complex Roots.*

Theorem 3 (Complex Conjugate Roots)**.** *Suppose that the indicial polynomial of $ax^2y''+bxy'+cy = 0$ has two complex conjugate roots: $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$. Then the general solution to $ax^2y'' + bxy' + cy = 0$ is*

$$y_G = c_1x^\alpha \cos(\beta \ln x) + c_2x^\alpha \sin(\beta \ln x).$$

How did we arrive at $y_1 = x^\alpha \cos(\beta \ln x)$ and $y_2 = x^\alpha \sin(\beta \ln x)$? In the following way:

A priori we have

$$y_3 = x^{\alpha+i\beta}$$

and

$$y_4 = x^{\alpha-i\beta}.$$

Start by rewriting them in the following form:

$$y_3 = x^\alpha e^{i\beta \ln x}$$

and

$$y_4 = x^\alpha e^{-i\beta \ln x}.$$

This may indeed seem strange, but by applying Euler's formula we have:

$$y_3 = x^\alpha(\cos(\beta \ln x) + i \sin(\beta \ln x))$$

and

$$y_4 = x^\alpha(\cos(\beta \ln x) - i \sin(\beta \ln x)).$$

Now taking the following complex linear combinations, we arrive at the y_1 and y_2 above:

$$y_1 = \frac{1}{2}(y_3 + y_4)$$

and

$$y_2 = \frac{1}{2i}(y_3 - y_4).$$

Example 3. *Solve the equation*

$$2x^2y'' - 2xy' + 20 = 0.$$

Solution. *The indicial equation is*

$$2m(m-1) - 2m + 20 = 2m^2 - 4m + 20 = 0,$$

which has roots:

$$m = \frac{4 \pm \sqrt{16-160}}{4} = \frac{4 \pm \sqrt{-144}}{4} = \frac{4 \pm 12i}{4} = 1 \pm 3i.$$

So the general solution is

$$y_G = c_1 x\cos(3\ln x) + c_2 x\sin(3\ln x).$$

◇

Exercises. Solve the given equation, if there is an initial value, find the solution of the IVP. Assume that $x > 0$ in all of the problems.

(1) $x^2y'' - 2y = 0$

(2) $x^2y'' + 4xy' + 2y = 0; y(1) = 1, y'(1) = 2$

(3) $x^2y'' + 3xy' - 3y = 0$

(4) $x^2y'' - 3xy' + 4y = 0; y(1) = 2, y'(1) = 1$

(5) $3xy'' + 2y' = 0$

(6) $4x^2y'' + y = 0$

(7) $x^2y'' + xy' + 4y = 0; y(1) = 1, y'(1) = 4$

(8) $x^2y'' - 5xy' + 13y = 0$

Challenge Problems:

(9) Show that the change of variable $t = ax + b$ transforms the equation

$$b_0(ax+b)^2y'' + b_1(ax+b)y' + b_2y = 0$$

into a Cauchy-Euler equation.

(10) Use the result of Exercise 9 to find the general solution of the given equation:
  (a) $(x-3)^2y'' + 3(x-3)y' + y = 0, x > 3$
  (b) $(2x+1)^2y'' + 4(2x+1)y' - 24y = 0, x > -\frac{1}{2}$

2.6. The Method of Undetermined Coefficients.

The method of undetermined coefficients is useful for finding solution of the equation:

$$ay'' + by' + cy = f(x) \tag{2.7}$$

where $f(x)$ is a sum of functions of the form $P_n(x)e^{\alpha x}\sin\beta x$ or $Q_m(x)e^{\alpha x}\cos\beta x$, where α and β can be any real number ($\beta \geq 0$), n and m are integers greater than or equal to zero, and P_n and Q_m are polynomials of degree n and m respectively.

Rather than try to derive all of the choices, here we will just exemplify and collect the different cases we can solve: To sum up the method of undetermined coefficients, consider the following table for the equation $ay''+by'+cy = f(x)$:

If $f(x) =$	Guess $y_p =$
(1) $P_n(x)$	$x^s(A_nx^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0)$
(2) $P_n(x)e^{\alpha x}$	$x^s(A_nx^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0)e^{\alpha x}$
(3) $P_n(x)e^{\alpha x}\sin\beta x$ and/or $Q_m(x)e^{\alpha x}\cos\beta x$	$x^s e^{\alpha x}[(A_kx^k + A_{k-1}x^{k-1} + \cdots + A_1x + A_0)\cos\beta x$ $+(B_kx^k + B_{k-1}x^{k-1} + \cdots + B_1x + B_0)\sin\beta x]$

where $k = \max\{n, m\}$ if and s is determined as follows:

(1) s equals the number of times 0 is a root of the characteristic equation.
(2) s equals the number of times α is a root of the characteristic equation.
(3) $s = 1$ if $\alpha + i\beta$ is a root of the characteristic equation and 0 otherwise.

(Notice that $s = 0$, 1, or 2.)

Using the table above combined with the principle of superposition, we can solve any of the equations mentioned at the beginning of this section. The general method is to guess the appropriate, plug it in, and equate coefficients with the nonhomogeneous part, $f(x)$.

The motivation behind each of the choices is as follows:

(1) Since f is just a polynomial and the equation is just a linear combination of derivatives of y, the solution will be a polynomial. When we guess a particular solution, it must be a polynomial of the same degree as f, and we must have a term of every power less than that in order to compensate for the fact that differentiation lowers the degree of a polynomial by one.
(2) The same as one, except that now we must include an exponential term since that is multiplied onto the polynomial. We don't have to add anything extra (unless we need $s = 1, 2$) since when we differentiate $e^{\alpha x}$ all that happens is it gets multiplied by α.
(3) If either a sine term or a cosine term show up in f (one, the other, or both), we need to include both sin and cos in our guess since their derivates alternate between each other.

The motivation for the x^s is to ensure that we can equate coefficients properly.

We have already covered item (1) in the table above in Section 2.3, so now let's see an example of the other two. Keep in mind that a polynomial of degree 0 is just a constant. First an example of item (2).

Example 1. *Solve the differential equation*

$$y'' - 4y' + 3y = e^{3x}(x + 1).$$

Solution. *A fundamental set of solutions for the associated homogeneous equation is $\{e^{3x}, e^x\}$, so by the table above we have that $s = 1$ and we should guess that $y_p = x(Ax + B)e^{3x} = (Ax^2 + Bx)e^{3x}$. Now we need to take the first and second derivatives of y_p:*

$$\begin{aligned}
y_p' &= (2Ax + B)e^{3x} + (3Ax^2 + 3Bx)e^{3x} \\
&= (3Ax^2 + (2A + 3B)x + B)e^{3x} \\
y_p'' &= (6Ax + (2A + 3B))e^{3x} + (9Ax^2 + (6A + 9B)x + 3B)e^{3x} \\
&= (9Ax^2 + (12A + 3B)x + (2A + 3B))e^{3x}
\end{aligned}$$

Plug these into the differential equation:

$$\begin{aligned} y_p'' - 4y_p' + 3y_p &= (9Ax^2 + (12A+3B)x + (2A+3B))e^{3x} - 4((3Ax^2 + (2A+3B)x + B)e^{3x}) + 3((Ax^2+Bx)e^{3x}) \\ &= (9Ax^2 + 12Ax + 9Bx + 2A + 6B - 12Ax^2 - 8Ax - 12Bx - 4B + 3Ax^2 + 3Bx)e^{3x} \\ &= (4Ax + (2A+2B))e^{3x} \\ &= (x+1)e^{3x} \end{aligned}$$

Equating coefficients yields the system of equations:

$$\begin{cases} 4A & = 1 \\ 2A + 2B & = 1 \end{cases}$$

which has the solution $A = \frac{1}{4}$ *and* $B = \frac{1}{4}$ *so that*

$$y_p = \frac{1}{4}x^2e^{3x} + \frac{1}{4}xe^{3x}.$$

Thus our general solution is:

$$y_G = c_1e^x + c_2e^{3x} + \frac{1}{4}x^2e^{3x} + \frac{1}{4}xe^{3x}.$$

◇

Maybe you noticed in this example, but whenever we have an exponential term in f, it makes differentiation quite annoying, and it ends up getting canceled out before we solve for the unknown coefficients. One way to make this method more efficient is to first make the guess that $y_p = u_pe^{\alpha x}$, which is what it will look like anyway. Plug this into the differential equation to get rid of the exponential term, then just solve the equation to find u_p. Let's see this method in an example of (3) in the chart above:

Example 2. *Solve the differential equation*

$$y'' - 2y' + 5y = e^{-x}\cos x.$$

Solution. *A fundamental set of solutions to this differential equation is:* $\{e^x\cos 2x, e^x\sin 2x\}$*, so* $s = 0$ *and we should, by the table, guess that* $y_p = (A\cos x + B\sin x)e^{-x}$*, but let's instead first try* $y_p = ue^{-x}$*. So start by differentiating* y_p*:*

$$y_p' = u'e^{-x} - ue^{-x}$$

and

$$y_p'' = u''e^{-x} - 2u'e^{-x} + ue^{-x}.$$

Plugging this into the equation we get:

$$\begin{aligned} y_p'' - 2y_p' + 5y_p &= u''e^{-x} - 2u'e^{-x} + ue^{-x} - 2(u'e^{-x} - ue^{-x}) + 5ue^{-x} \\ &= (u'' - 4u' + 8u)e^{-x} \\ &= e^{-x}\cos x \end{aligned}$$

Which reduces to the equation

$$u'' - 4u' + 8u = \cos x$$

Since $\cos x$ *and* $\sin x$ *are not solutions to the corresponding homogeneous differential equation for* u*, we can use the chart above to guess that*

$$u_p = A\cos x + B\sin x$$

and plug it in:

$$-A\cos x - B\sin x - 4(-A\sin x + B\cos x) + 8(A\cos x + B\sin x) = (7A - 4B)\cos x + (4A + 7B)\sin x = \cos x.$$

This gives us the system of equations

$$\begin{cases} 7A - 4B & = 1 \\ 4A + 7B & = 0 \end{cases}$$

which has the solution $A = \frac{7}{65}$ *and* $B = -\frac{4}{65}$*. Thus our particular solution should be* $y_p = u_pe^{-x} = \frac{7}{65}e^{-x}\cos x - \frac{4}{65}e^{-x}\sin x$*, which it indeed is (you should check this to convince yourself). Therefore our general solution is:*

$$y_G = c_1e^x\cos 2x + c_2e^x\sin 2x + \frac{7}{65}e^{-x}\cos x - \frac{4}{65}e^{-x}\sin x.$$

◇

Exercises.

Find the general solution to the given differential equation. If there is an initial value, find the solution to the IVP.

(1) $y'' - 2y' - 3y = e^x(3x - 8)$

(2) $y'' + 4y = e^{-x}(5x^2 - 4x + 7)$

(3) $y'' - 4y' - 5y = -6xe^{-x}$

(4) $y'' - 4y' - 5y = 9e^{2x}(x + 1)$; $y(0) = 0, y'(0) = -10$

(5) $y'' - 3y' - 10y = 7e^{-2x}$; $y(0) = 1, y'(0) = -17$

(6) $y'' + y' + y = xe^x + e^{-x}(2x + 10$

(7) $y'' - 8y' + 16y = 6xe^{4x} + 16x^2 + 16x + 2$

(8) $y'' + 3y' + 2y = 7\cos x - \sin x$

(9) $y'' + y = (8x - 4)\cos x + (-4x + 8)\sin x$

(10) $y'' + 2y' + y = e^x(6\cos x + 17\sin x)$

(11) $y'' + 3y' - 2y = e^{-2x}[(20x + 4)\cos 3x + (-32x + 26)\sin 3x]$

(12) $y'' + 9y = \sin 3x + \cos 2x$

(13) $y'' + 6y' + 10y = -40e^x \sin x$; $y(0) = 2, y'(0) = -3$

(14) $y'' - 2y' + 2y = 4xe^x \cos x + xe^{-x} + x^2 + 1$

2.7. Variation of Parameters.

In this section we will study the powerful method of Variation of Parameters for finding general solutions to non-homogeneous equations. To use this method, we need to know a fundamental set of solutions for our equation. Why would we want to do this when it seems that reduction of order is better since it only requires us to know one of the homogeneous solutions? One reason is that it is usually much simpler to do than reduction of order. Another is it generalizes very easily to higher order linear differential equations, unlike reduction of order.

Suppose that we have the differential equation

$$p(x)y'' + q(x)y' + r(x)y = f(x),$$

and that we know the two homogeneous solutions y_1 and y_2. With this method we will search for solutions of the form $y = u_1y_1 + u_2y_2$. Let's differentiate this

$$y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'.$$

Since we have two unknown functions ($u_1$andu_2) we need two equations to solve for both of them, so we create the first equation by demanding that

$$u_1'y_1 + u_2'y_2 = 0.$$

This is good because, not only does it give us one constraint on u_1 and u_2, it also simplifies y':

$$y' = u_1y_1' + u_2y_2'.$$

Now take the derivative again:

$$y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$$

and plug y, y', and y'' into the differential equation and group by derivatives of u_1 and u_2 to get:

$$\begin{aligned} p(x)y'' + q(x)y' + r(x)y &= p(x)(u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + q(x)(u_1y_1' + u_2y_2') + r(x)(u_1y_1 + u_2y_2) \\ &= p(x)u_1'y_1' + p(x)u_2'y_2' + (p(x)y_1'' + q(x)y_1' + r(x)y_1)u_1 + (p(x)y_2'' + q(x)y_2' + r(x)y_2)u_2 \\ &= p(x)u_1'y_1' + p(x)u_2'y_2' + 0u_1 + 0u_2 \\ &= p(x)u_1'y_1' + p(x)u_2'y_2' = f(x) \end{aligned}$$

So the equation reduces to:

$$u_1'y_1' + u_2'y_2' = \frac{f(x)}{p(x)}.$$

This equation gives us another restriction on u_1 and u_2! So now we have a system of equations to determine u_1 and u_2 with:

$$\left\{ \begin{array}{ccccc} u_1'y_1 & + & u_2'y_2 & = & 0 \\ u_1'y_1' & + & u_2'y_2' & = & \dfrac{f(x)}{p(x)} \end{array} \right. \tag{2.8}$$

Using this system of equations, we can solve for u_1 and u_2 (leaving the integration constants in!) and get the general solution to our differential equation:

$$y_G = u_1y_1 + u_2y_2.$$

There is one more thing that I must convince you of, that is that we can always solve for u_1 and u_2 in this system of equations. To see this, start by multiplying the first equation by y_2' and the second equation by y_2:

$$\left\{ \begin{array}{ccccc} u_1'y_1y_2' & + & u_2'y_2y_2' & = & 0 \\ u_1'y_2y_1' & + & u_2'y_2y_2' & = & \dfrac{f(x)y_2}{p(x)} \end{array} \right.$$

now subtract the second equation from the first and get:

$$u_1'(y_1y_2' - y_2y_1') = -\frac{f(x)y_2}{p(x)}$$

but notice that $y_1y_2' - y_2y_1' = W(x; y_1, y_2)$, thus:

$$u_1' = -\frac{f(x)y_2}{p(x)W(x; y_1, y_2)}$$

so that

$$u_1 = -\int \frac{f(x)y_2}{p(x)W(x; y_1, y_2)}dx.$$

Similarly you can show that

$$u_2' = \frac{f(x)y_1}{p(x)W(x; y_1, y_2)}$$

so that

$$u_2 = \int \frac{f(x)y_1}{p(x)W(x; y_1, y_2)} dx.$$

Remember to keep the constants of integration in these two integrals (and to call them different names, preferably c_1 for u_1 and c_2 for u_2) so that when we write $y = u_1y_1 + u_2y_2$ we get the general solution!

It is not recommended that you memorize the equations for u_1 and u_2, as they were derived merely to show that you can always find them. However, what you should do when you want to use variation of parameters is the following:

(1) Find y_1 and y_2 for you differential equation.
(2) Write down the system (2.8) with your y_1 and y_2 plugged in.
(3) Solve for u_1' or u_2', and use whichever you found first to find the second.
(4) Integrate u_1' and u_2' to find u_1 and u_2, keeping the integration constants if you are trying to find the general solution.
(5) Write down and simplify the equation $y_G = u_1y_1 + u_2y_2$ (or $y_p = u_1y_1 + u_2y_2$ if you don't keep the constants of integration).

Example 1. *Solve the differential equation*

$$y'' + 4y = \sin 2x \sec^2 2x.$$

Solution. *A fundamental set of solutions to this problem is* $\{\cos 2x, \sin 2x\}$*, so now let's write down the system of equations:*

$$\begin{cases} u_1' \cos 2x & + & u_2' \sin 2x & = & 0 \\ -2u_1' \sin 2x & + & 2u_2' \cos 2x & = & \sin 2x \sec^2 2x \end{cases}$$

Multiply the first equation by $2\sin 2x$ *and the second by* $\cos 2x$*, then add them together to get:*

$$2u_2' = \sin 2x \sec 2x = \tan 2x$$

integrating both sides and dividing by 2 we get:

$$u_2 = -\frac{1}{4} \ln|\cos 2x| + c_2.$$

Plug the equation for $2u_2'$ *into the second equation in the system above to get:*

$$-2u_1' \sin 2x + \sin 2x = \sin 2x \sec^2 2x.$$

Dividing by $\sin 2x$ *and isolating* u_1' *we get:*

$$u_1' = \frac{1}{2} - \frac{1}{2}\sec^2 2x$$

and integrating we get:

$$u_1 = \frac{1}{2}x - \frac{1}{4}\tan 2x + c_1.$$

Thus our general solution is

$$\begin{aligned} y_G &= u_1y_1 + u_2y_2 = \left(\frac{1}{2}x - \frac{1}{4}\tan 2x + c_1\right)\cos 2x + \left(-\frac{1}{4}\ln|\cos 2x| + c_2\right)\sin 2x \\ &= \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + c_1\cos 2x - \frac{1}{4}\sin 2x \ln|\cos 2x| + c_2 \sin 2x, \end{aligned}$$

and after some simplification:

$$y_G = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\sin 2x \ln|\cos 2x| + \frac{1}{2}x\cos 2x.$$

◇

Exercises.

Use variation of parameters to find the general solution given a fundamental set of solutions to the differential equation.

(1) $xy'' + (2-2x)y' + (x-2)y = e^{2x}$; $\{e^x, x^{-1}e^x\}$

(2) $(\sin x)y'' + (2\sin x - \cos x)y' + (\sin x - \cos x)y = e^{-x}$; $\{e^{-x}, e^{-x}\cos x\}$

(3) $x^2y'' + xy' - y = 2x^2 + 2$; $\left\{x, \dfrac{1}{x}\right\}$

(4) $x^2y'' - xy' - 3y = x^{\frac{3}{2}}$; $\left\{\dfrac{1}{x}, x^3\right\}$

Use variation of parameters to find the general solution.

(5) $y'' + 9y = \tan 3x$

(6) $y'' - 3y' + 2y = \dfrac{4}{1+e^{-x}}$

(7) $y'' - 2y' + 2y = 2e^x \sec x$

(8) $y'' - 2y' + y = 14x^{\frac{3}{2}}e^x$

(9) $y'' - y = \dfrac{2}{e^x + 1}$

(10) $y'' - 2y' + y = \dfrac{1}{x}e^x$

(11) $y'' - 2y' + y = 4e^{-x}\ln x$

(12) $y'' + y = \sec x \tan x$

(13) $x^2y'' - 2xy' + 2y = x^3e^x$

2.8. Some Non-Linear Second Order Equations.

Recall that in its most general form, a second order differential equation is of the form:

$$F(x, y, y', y'') = 0.$$

In general, non-linear differential equations can become quite nasty very fast. We usually do not have a nice set of fundamental set of solutions; in fact, we usually have a multitude of solutions that seem to have no relation whatsoever. These equations may become easier under certain circumstances. In this section we will study the two cases in which y is not an input of F and when x is not an input of F. In these two cases, we can reduce solving the second order equation down to solving two first order differential equations.

2.8.1. *Missing Dependent Variable.*

Suppose we had a differential equation of the form:

$$G(x, y', y'') = 0.$$

In this case, the dependent variable y is not an input of G. (Recall that the method of reduction of order yields an equation of this type). Using the substitution $u = y'$, this equation becomes:

$$G(x, u, u') = 0$$

which is a first order differential equation. If we can solve this equation for u (there is no guarantee that we can!), then to get y we simply integrate u, since $y' = u$.

Example 1. *Solve the equation*

$$x\frac{d^2y}{dx^2} = 2\left[\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx}\right].$$

Solution. *First let's start by letting $v = y'$, then the equation becomes:*

$$xv' = 2(v^2 - v) = 2v(v-1)$$

suggesting that $v_1 = 0$ and $v_2 = 1$ are constant solutions. Thus since $v = y'$ we have that $y_1 = \int v_1\,dx = c$ and $y_2 \int v_2\,dx = x + c$ are solutions. Now the differential equation for v is separable, so separate it into:

$$\frac{1}{v(v-1)}dv = \frac{2}{x}dx$$

using partial fractions to rewrite the equation we get:

$$\left(\frac{1}{v-1} - \frac{1}{v}\right)dv = \frac{2}{x}dx$$

now integrate:

$$\ln|v-1| - \ln|v| = \ln x^2 + c$$

and solve for v:

$$\begin{aligned} & \ln\left|\tfrac{v-1}{v}\right| &=& \ln x^2 + c \\ \Longrightarrow\quad & \tfrac{v-1}{v} &=& cx^2 \\ \Longrightarrow\quad & v &=& \tfrac{1}{1-cx^2} \end{aligned}$$

Then in order to integrate v to solve for y, we need to consider the different signs of c:

$c = 0$) *If $c = 0$ then $v = 1$ which is handled above.*

$c > 0$) *If $c > 0$ let $c = a^2$, then*

$$v_3 = \frac{1}{1-(ax)^2} = \frac{1}{(1+ax)(1-ax)} = \frac{\frac{1}{2}}{1+ax} + \frac{\frac{1}{2}}{1-ax}$$

which implies that

$$y_3 = \frac{1}{2}\frac{1}{a}\ln|1+ax| + \frac{1}{2}\frac{1}{(-a)}\ln|1-ax| + k = \frac{1}{2a}\ln\left|\frac{1+ax}{1-ax}\right| + k.$$

$c < 0$) *If $c < 0$ then let $c = -b^2$, then*

$$v_4 = \frac{1}{1+(bx)^2}$$

so that when we integrate we get

$$y_4 = \frac{1}{b}\arctan bx + k.$$

Thus the solutions to the equation are, where c is an arbitrary constant, and a is any nonzero number:

$$\begin{aligned} y_1 &= c \\ y_2 &= x + c \\ y_3 &= \frac{1}{2a}\ln\left|\frac{1+ax}{1-ax}\right| + c \\ y_4 &= \frac{1}{a}\arctan ax + c \end{aligned}$$

(Note that the a and c here are not the same as above.)

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2.8.2. *Missing Independent Variable.*

Now suppose we had a differential equation of the form:

$$H(y, y', y'') = 0.$$

Here the independent variable x is missing. Suppose that y is a solution to this differential equation. Then let $v = \frac{dy}{dx}$. On an interval where y is a strictly monotone function, we can regard x as a function of y (since the function is invertible on this interval) and we can write (by the chain rule):

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}$$

which leads to

$$H\left(y, v, v\frac{dv}{dy}\right) = 0$$

which is a first order differenial equation. Upon finding v, we can then integrate it to find y.

Example 2. *Solve the equation*

$$y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx}.$$

Solution. *Start by making the substitution* $v = \frac{dy}{dx}$*. Then* $\frac{dv}{dx} = v\frac{dv}{dy}$*. Substitute these into the equation and get:*

$$yv\frac{dv}{dy} = v^2 + 2v = v(v+2)$$

from which it is evident that $v_1 = 0$ *and* $v_2 = -2$ *are solutions so that* $y_1 = c$ *and* $y_2 = -2x + c$ *are solutions to the original equation. The differential equation for* v *is separable, to see this first divide by* v *(we don't have to worry about whether or not* $v = 0$ *here since we have already established that* $v = 0$ *is a solution (see* v_1 *above)):*

$$y\frac{dv}{dy} = (v+2)$$

which separates into

$$\frac{1}{v+2}dv = \frac{1}{y}dy.$$

Integrating both sides, we arrive at:

$$\ln|v+2| = \ln|y| + c$$

which gives

$$v = cy - 2.$$

If $c = 0$ *above we have no worries since then we would have* $v = -2$ *which we already handled above. So assume* $c \neq 0$*, then the equation can be written in the form, replacing* v *with* $\frac{dy}{dx}$*:*

$$\frac{dy}{dx} - cy = -2$$

which is a first order linear equation for y*. Solving this we get the solution*

$$y_3 = \frac{1}{c}\left(ke^{cx} + 2\right).$$

So collecting all of the solutions together we have:

$$\begin{aligned} y_1 &= c \\ y_2 &= -2x + c \\ y_3 &= \frac{1}{c}\left(ke^{cx} + 2\right) \end{aligned}$$

where c and k are arbitrary constants (in y_3, $c \neq 0$).

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Exercises.

Solve the differential equation:

(1) $t\dfrac{d^2x}{dt^2} = 2\dfrac{dx}{dt} + 2$

(2) $2x\dfrac{dy}{dx}\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2 + 1$

(3) $xe^{\frac{dx}{dt}}\dfrac{d^2x}{dt^2} = e^{\frac{dx}{dt}} - 1$

(4) $y\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$

(5) $\dfrac{d^2y}{dx^2} + e^{-y}\dfrac{dy}{dx}$

(6) $(x^2 + 1)\dfrac{d^2y}{dx^2} = 2x\left(\dfrac{dy}{dx}\right)^2$

Challenge Problem:

(7) $x\dfrac{d^3y}{dx^3} = 2\dfrac{d^2y}{dx^2}$

2.9. **Boundary Value Problems.**

Exercises.

2.10. Additional Exercises.

Determine whether the given set of functions in linearly independent on $(-\infty, \infty)$:

(1) $\{\sin 3x, \cos 3x\}$

(2) $\{3x, 4x\}$

(3) $\{x^2, x\}$

(4) $\{x^2, 5\}$

(5) $\{e^{2x}, e^{3x}\}$

(6) $\{3\cos 6x, 1 + \cos 3x\}$

(7) $\{3e^{2x}, 5e^{2x}\}$

Solve the given equation:

(8) $y'' - y = 0$

(9) $y'' - 2y' + y = 0$

(10) $y'' + 2y' + 2y = 0$

(11) $y'' - 7y = 0$

(12) $y'' + 6y' + 9y = 0$

(13) $y'' - 3y' - 5y = 0$

(14) $x'' - 20x' + 64x = 0$

(15) $x'' + x' + 2x = 0$

(16) $u'' - 36u = 0$

(17) $\dfrac{d^2Q}{dt^2} - 5\dfrac{dQ}{dt} + 7Q = 0$

(18) $x'' - 10x' + 25x = 0$

(19) $\dfrac{d^2P}{dt^2} - 7\dfrac{dP}{dt} + 9P = 0$

(20) $\dfrac{d^2N}{dx^2} + 5\dfrac{dN}{dx} + 24N = 0$

(21) $\dfrac{d^2T}{d\theta^2} + 30\dfrac{dT}{d\theta} + 225T = 0$

(22) $\dfrac{d^2R}{d\theta^2} + 5\dfrac{dR}{d\theta} = 0$

(23) $y'' - 2y' + y = x^2 + 1$

(24) $y'' - 2y' + y = 3e^{2x}$

(25) $y'' - 2y' + y = 4\cos x$

(26) $y'' - 2y' + y = 3e^x$

(27) $y'' - 2y' + y = xe^x$

(28) $y'' - 2y' + y = \dfrac{1}{x^5}e^x$

(29) $y'' + y = \sec x$

(30) $y'' - y' - 2y = e^{3x}$

(31) $y'' - 7y' = -3$

(32) $y'' + \frac{1}{x}y' - \frac{1}{x^2}y = \ln x$

(33) $x^2y'' + 7xy' + 9y = 0$

(34) $x^2y'' - 2y = x^3e^{-x}$

(35) $x^2y'' + xy' + y = 0$

(36) $x^2y'' - 4xy' - 6y = 0$

(37) $y'' + y = \tan t$

(38) $y'' + 9y = 9\sec^2 3t$

(39) $4y'' + y = 2\sec\left(\frac{t}{2}\right)$

(40) $t^2y'' - 2y = 3t^2 - 1$

(41) $x^2y'' - 3xy + 4y = x^2\ln x$

Solve the given differential equation. You are given at least one of the homogeneous solutions.

(42) $(1-x)y'' + xy' - y = 2(t-1)^2e^{-t}$; $y_1 = e^t$

(43) $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) = 3x^{\frac{3}{2}}\sin x$; $y_1 = x^{-\frac{1}{2}}\sin x, y_2 = x^{-\frac{1}{2}}\cos x$

(44) $y'' - (1+t)y' + y = t^2e^{2t}$; $y_1 = 1+t, y_2 = e^t$

(45) $t^2y'' - t(t+2)y' + (t+2)y = 0$; $y_1 = t$

(46) $xy'' - y' + 4x^3y = 0$; $y_1 = \sin x^2$

(47) $x^2y'' - \left(x - \frac{3}{16}\right)y = 0$; $y_1 = x^{\frac{1}{4}}e^{2\sqrt{x}}$

Solve the given initial value problem.

(48) $y'' - y' - 2y = e^{3x}$; $y(0) = 1, y'(0) = 2$

(49) $y'' - y' - 2y = 0$; $y(0) = y'(0) = 2$

(50) $y'' + 4y = \sin^2 2x$; $y(\pi) = y'(\pi) = 0$

(51) $y'' + 3y' + 2y = \sin 2x + \cos 2x$; $y(0) = 0, y'(0) = 1$

3. Higher-Order Differential Equations

3.1. General nth order Linear Differential Equations.

Exercises.

3.2. **The Method of Undetermined Coefficients.**

Exercises.

3.3. **Variation of Parameters.**

Exercises.

3.4. **Additional Exercises.**

4. Some Applications of Differential Equations

4.1. Orthogonal Trajectories.

In this section we will explore how to find functions orthogonal to a given family of functions.

Consider the equation

$$y = cx$$

where c is an arbitrary constant. This equation describes a family of lines through the origin. The slope of a given member of the family is its derivative:

$$y' = c.$$

If we solve for c in the original equation (which we can do since we picked a specific member of the family) we get:

$$c = \frac{y}{x}.$$

With the exception of the origin, every point (x, y) in the plane has exactly one member of the family passing though it. At the point (x, y) in the plane, the slope of the member of family member passing though it is given by

$$y' = c = \frac{y}{x}.$$

Suppose we wanted the equation of a graph passing though that point perpendicular to every member of the family. How would we do this? Recall the definition of perpendicular slope from Algebra 1:

If m is our original slope, the perpendicular slope is given by $m_{\perp} = -\frac{1}{m}$.

We will do a similar thing to get an orthogonal function:
Since our original slope is:

$$y' = \frac{y}{x}$$

our perpendicular slope is:

$$y' = -\frac{x}{y}$$

which is a separable equation that can be solved to get:

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + k$$

which can be rewritten as:

$$x^2 + y^2 = k.$$

Thus the orthogonal family of functions are concentric circles of every radius centered at the origin!

Exercises. Find the orthogonal trajectories of each given family. Also sketch several members of each family.

(1) $y = e^x + c$

(2) $y = ce^x$

(3) $y = \arctan x + c$

(4) $x^2 + \frac{1}{4}y^2 = c$

(5) $x^2 + (y - c)^2 = c^2$

(6) $x^2 - y^2 = 2cx$

4.2. Mixing Problems.

Suppose that we have a tank that initially has V_0 liters of a well-mixed solution containing Q_0 grams of salt. Suppose that we pump in a salt solution of concentration $q_{in}\dfrac{g}{liter}$ at a rate of $r_{in}\dfrac{liter}{min}$. Also suppose that we let the (well-mixed) solution drain at a rate of $r_{out}\dfrac{liter}{min}$. We would like to know the amount of salt (in grams) in the tank at any time t. How would we do this? We, given the information above, know the rate at which the amount of salt is changing with respect to time. If we let $Q(t)$ be the the amount of salt in the tank at time t, the equation that gives the rate of change of the amount of salt is:

$$Q'(t) = r_{in}q_{in} - r_{out}q_{out}; Q(0) = Q_0 \tag{4.1}$$

where q_{out} is the concentration of salt in the solution leaving the tank. So the only thing we are missing is q_{out}. This is given by taking the amount of salt in the tank at time t and dividing it by the volume of solution in the tank at the same time, i.e.:

$$q_{out} = \frac{Q(t)}{V(t)},$$

where $V(t) = V_0 + (r_{in} - r_{out})t$ is the volume of the tank at time t. Substituting this into (4.1) we get:

$$Q' = r_{in}q_{in} - r_{out}\frac{Q(t)}{V(t)}; Q(0) = Q_0,$$

which is a linear IVP whose solution is the amount of salt in the tank at time t.

Let's see an example of this.

Example 1. *A tank originally contains 100 liters of fresh water (i.e. contains* 0 *grams of salt). Then water containing* $\frac{1}{2}$ *grams of salt per liter is pumped into the tank at a rate of* $2\frac{liters}{minute}$*, and the well-stirred mixture leaves the tank at the same rate. Find the amount of salt in the tank at time t after the process started.*

Solution. *Suppose that* $Q(t)$ *is the amount of salt in the tank at time* t*. Let's first begin by finding* r_{in}*,* r_{out}*,* q_{in}*,* Q_0*,* $V(t)$*, and* q_{out}*:*

$$\begin{aligned} r_{in} &= 2 \\ r_{out} &= 2 \\ q_{in} &= \frac{1}{2} \\ Q_0 &= 0 \\ V(t) &= 100 + (2-2)t = 100 \\ q_{out} &= \frac{Q(t)}{V(t)} = \frac{1}{100}Q. \end{aligned}$$

Now let's set up the equation for Q'*:*

$$Q' = (2)\left(\frac{1}{2}\right) - (2)\left(\frac{1}{100}Q\right); Q(0) = 0.$$

Simplifying and rewriting this, we get:

$$Q' + \frac{1}{50}Q = 1; Q(0) = 0.$$

The general solution to this equation is:

$$Q(t) = ce^{-\frac{1}{50}t} + 50$$

and using the initial value we get $c = -50$ *so that our solution is:*

$$Q(t) = -50e^{-\frac{1}{50}t} + 50.$$

This function gives the amount of salt in the tank at time t*. Notice that if we take the limit as* $t \to \infty$ *we get that there is* 50 *grams of salt in the tank. This is consistent with the fact that the solution that is being poured in has a salt concentration of* 50% *and that eventually an equilibrium point will be reached.*

◇

Exercises.

(1) A tank initially contains 120 liters of pure water. A mixture containing a concentration of $\gamma \frac{g}{liter}$ of salt enters the tank at a rate of $3\frac{liters}{min}$, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$.

(2) A tank with a capacity of 500 liters originally contains 200 liters of water with 100 kilograms of salt in the solution. Water containing 1 kg of salt per liter is entering at a rate of $3\frac{liter}{min}$, and the mixture is allowed to flow out of the tank at a rate of $2\frac{liter}{min}$. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow. Find the concentration (in kilograms per liter) of salt in the tank at the instant it starts to overflow.

(3) A tank initially holds 25 liters of water. Alcohol with a concentration of $5\frac{g}{L}$ enters at the rate of $2\frac{liter}{min}$ and the mixture leaves at the rate of $1\frac{liter}{min}$. What will be the concentration of alcohol when 50 liters of fluid is in the tank?

(4) A tank initially contains 50 liters of a solution that holds 30 grams of a chemical. Water runs into the tank at the rate of $3\frac{liter}{min}$ and the mixture runs out at the rate of $2\frac{liter}{min}$. After how long will there be 25 grams of the chemical in the tank?

4.3. **Radioactive Decay.**

Exercises.

4.4. Compound Interest.

Exercises.

4.5. **Topics in Mechanics.**

4.5.1. *Some Useful Stuff.*

Amplitude-Phase Form

Suppose we had

$$y = c_1 \cos \omega t + c_2 \sin \omega t$$

as a solution to a differential equation. Notice that this describes a system that is in a simple harmonic motion, or oscillation. Now let's define a few things:

$$R = \sqrt{c_1^2 + c_2^2},$$
$$c_1 = R \cos \varphi,$$

and

$$c_2 = R \sin \varphi,$$

where $\varphi \in [-\pi, \pi)$.

Then plugging these into the equation for y and using the identity $\cos(a-b) = \cos a \cos b + \sin a \sin b$ we get:

$$y = R(\cos \varphi \cos \omega t + \sin \varphi \cos \omega t) = R \cos(\omega t - \varphi).$$

The equation

$$y = R \cos(\omega t - \varphi) \tag{4.2}$$

is called the *Amplitude-Phase Form* of y. In the equation, R is the amplitude, φ is called the phase angle (measure in radians), and ω is the frequency.

Polar Coordinates

In the plane $\mathbb{R}^2$ we can define a new coordinate system that describes a point in the plane by its distance from the origin and the angle the line connecting the point to the origin makes with the positive x-axis. These coordinates are called *polar coordinates* and are written (r, θ), where r and θ are given by:

$$r = \sqrt{x^2 + y^2}$$

and

$$\theta = \arctan \frac{y}{x},$$

where x and y are the standard Cartesian coordinates of the point (r, θ). To go from polar to Cartesian, use the equations

$$x = r \cos \theta$$

and

$$y = r \sin \theta.$$

4.5.2. *Simple Spring Motion.*

In this section we will ignore air resistance and all of the other usual things to ignore. Suppose that we have a spring hanging from a support (see Figure 1 below) with spring constant k. Let L be the equilibrium length of the spring with no weight attached (Figure 1.a). Recall Hooke's Law which says that the force required to stretch a spring with spring constant k a displacement of x is

$$F = kx.$$

Now suppose that we attach a weight of mass m to the free end of the spring. Then the downward force of the mass is mg where g is the gravitational acceleration (which we will approximate as $g = 9.8 \frac{m}{s^2}$). Thus by Hooke's law and Newton's third law we have that:

$$mg = kd$$

where d is the distance that the weight has stretched the spring from its original rest length (Figure 1.b). Now suppose that we stretch the spring-mass system a length of x from its equilibrium position, then release it (Figure 1.c). We would like to model this motion.

By Newton's second law, differential equation that models the motion of the spring is given by:

$$my'' = mg - k(y + d)$$

where y, the vertical displacement of the mass, is a function of time. But since $mg = kd$ we have

$$my'' + ky = 0; y(0) = x, y'(0) = 0. \tag{4.3}$$

Note that if we give the mass an initial velocity of v_0 when released, the second initial value changes to $y'(0) = v_0$.

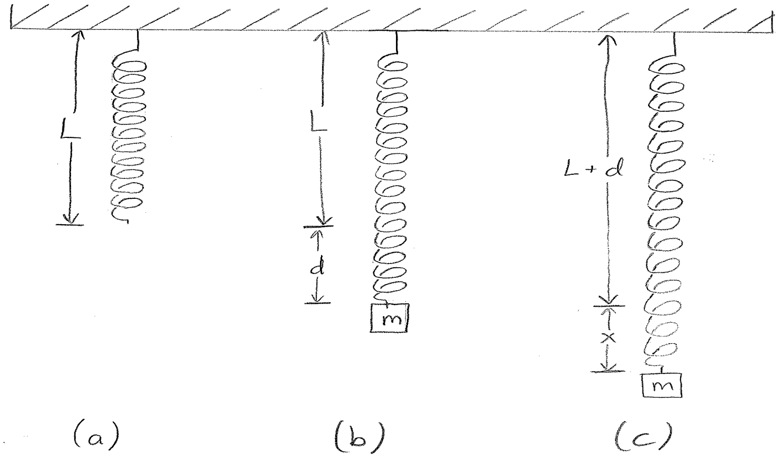


FIGURE 1. (a) A spring in at its natural length, (b) A spring with a weight of mass m attached, (c) A spring with a weight of mass m attached and displaced by a length x

Example 1. *An object of mass $\frac{1}{98}kg$ stretches a spring 5cm in equilibrium. Determine the equation of motion of the spring if the spring is stretched 10cm from equilibrium then released.*

Solution. *First we need to use Hooke's law to find the spring constant. Note that $m = \frac{1}{98}kg$ and $d = 5cm = .05m$ and set up the equation:*

$$mg = kd$$

then plug in what we know:

$$\left(\frac{1}{98}kg\right)\left(9.8\frac{m}{s^2}\right) = k(.05m)$$

which simplifies to:

$$\frac{1}{10}N = k(.05m)$$

and so:

$$k = 2\frac{kg \cdot m}{s^2}.$$

Now let's set up the differential equation (4.3) for the motion of the spring-mass system (note that the initial displacement is $x = -10cm = -0.1m$):

$$\frac{1}{98}y'' + 2y = 0; y(0) = -0.1, y'(0) = 0.$$

First we need to solve the equation:

$$y'' + 196y = 0.$$

The solution to this equation is:

$$y = c_1 \sin 14t + c_2 \cos 14t.$$

Now the initial values give:

$$y = -\frac{1}{10}\cos 14t$$

which is the equation of motion of the spring-mass system.

◇

4.5.3. *Motion Under a Central Force.*

Definition 1 (Central Force)**.** *A central force is a force whose magnitude at any point P not the origin depends only on the distance from P to the origin and whose direction at P is parallel to the line connecting P and the origin.*

An example of a central force is gravitation, and another is the electric force emitted by a point charge.

Assume that we are working with motion of an object in 3-dimensional space which is acted upon by a central force located at the origin. Notice that if the initial position vector and initial velocity vector of the object in consideration are parallel, then the motion of the particle is along the line connecting the initial position to the origin. Since this case is relatively uninteresting, we will consider the case in which they are not parallel. These two vectors determine a plane, and it is in this plane that the motion of the particle is taking place in. In this section we will derive how to find the path that the object takes through the plane, also known as the *orbit* of the object.

We will use polar coordinates to represent a central force. The central force can be written as:

$$\mathbf{F}(r,\theta) = f(r)(\cos\theta, \sin\theta) = (f(r)\cos\theta, f(r)\sin\theta).$$

Assume that f is a continuous function for all $r > 0$. Let's confirm that this is in fact a central force. First we need to check that the magnitude depends only on r, which is the distance from the object to the origin:

$$\|\mathbf{F}(r,\theta)\| = \sqrt{(f(r)\cos\theta)^2 + (f(r)\sin\theta)^2} = \sqrt{(f(r))^2[\cos^2\theta + \sin^2\theta]} = |f(r)| \cdot 1 = |f(r)|$$

which only depends on r. Now the direction of this force is from the origin to the point (r, θ) if $f(r) > 0$ and vice versa if $f(r) < 0$. Let's suppose that our object has mass m.

Recall Newton's second law of motion

$$\mathbf{F} = m\mathbf{a}.$$

Since our particle is in motion we can write r and θ as functions of time, i.e.

$$r = r(t) \text{ and } \theta = \theta(t),$$

and by using Newton's second law we have

$$\mathbf{F}(r,\theta) = m(r\cos\theta, r\sin\theta)'' = (m(r\cos\theta)'', m(r\sin\theta)'').$$

Exercises.

(1) An object stretches a spring $4cm$ in equilibrium. Find its displacement for $t > 0$ if it is initially displaced 36 cm above equilibrium and given a downward velocity of $25\frac{cm}{s}$.

(2) A spring with natural length $.5m$ has length $50.5cm$ with a mass of $2g$ suspended form it. The mass is initially displaced $1.5cm$ below equilibrium and released with zero velocity. Find its displacement for $t > 0$.

(3) An object stretches a spring $5cm$ in equilibrium. It is initially displaced $10cm$ above equilibrium and given an upward velocity of $.25\frac{m}{s}$. Find and graph is displacement for $t > 0$.

(4) A $10kg$ mass stretches a spring $70cm$ in equilibrium. Suppose that a $2kg$ mass is attached to the spring, initially displaced $25cm$ below equilibrium, and given an upward velocity of $2\frac{m}{s}$. Find its displacement for $t > 0$.

4.6. **Topics in Electricity and Magnetism.**

4.6.1. *The RLC Circuit.*

In this section we shall consider the RLC circuit which consists of a Resistor, Induction coil, and Capacitor connected in series as shown in Figure 2.

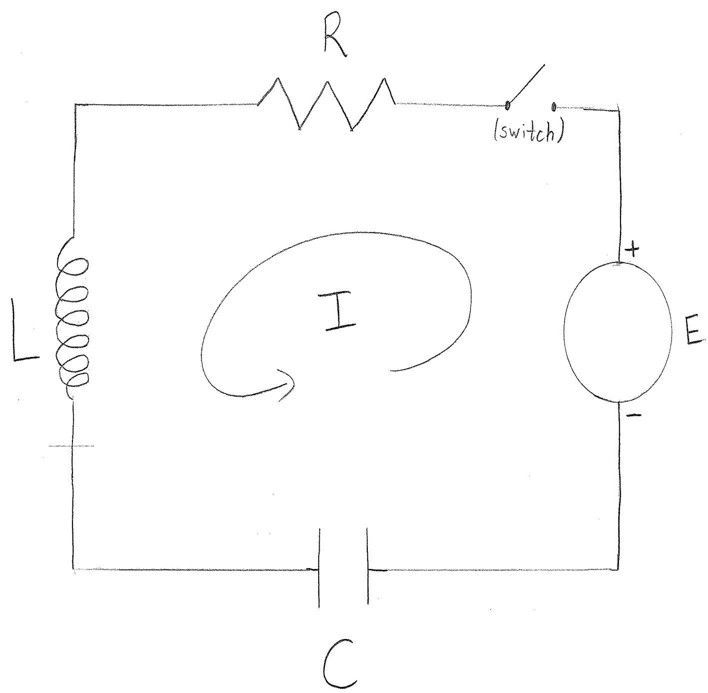


FIGURE 2. A simple RLC circuit. L is the induction coil, R is the resistor, C is capacitor, and E is a battery or generator.

Once the switch in the circuit is closed at time $t = 0$, current will begin to flow through the circuit. We will denote the value of the current at time t by $I(t)$. We will take current flowing in the direction of the arrow to be positive valued. Current begins to flow as a result of differences in electric potential created by the battery (or generator). Let's say that the battery (or generator) creates a potential difference of $E(t)$ which we will call the applied voltage. We will use the following convention for units:

amperes for the current I
volts for the voltage
ohms for the resistance R
henrys for the inductance L
farads for the capacitance C
and *coulombs* for the charge on the capacitor

Now we need to define the term *voltage drop*:

Definition 1 (Voltage Drop)**.** *The voltage drop across an element of a circuit is given by:*

Resistor) RI

Inductor) $L\dfrac{dI}{dt}$

Capacitor) $\dfrac{Q}{C}$ *where Q is the charge on the capacitor.*

We have the following relation between current and charge since current is defined to be the change in charge over time:

$$I = \frac{dQ}{dt}$$

and

$$Q = \int_0^t I(s)ds + Q_0$$

where Q_0 is the initial charge on the capacitor. Recall one of Kirchhoff's laws which says that the sum of the voltage drops around a circuit must be equal to the applied voltage. Thus by Kirchhoff's law we have that:

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t). \tag{4.4}$$

Let's differentiate this with respect to t to get:

$$LI'' + RI' + \frac{1}{C}I = E'(t) \tag{4.5}$$

Since we have that the initial current in the circuit is zero (since the circuit was not closed) we get one initial value

$$I(0) = 0.$$

Now we have to figure out what $I'(0)$ is. This will depend on what $E(t)$ is, and will be attained from equation (4.4).

We will consider two cases for $E(t)$ in this section:

(1) E is a battery and so $E(t) \equiv E_0$ a constant
(2) E is an alternating current (AC) generator with $E(t) = E_0 \cos \omega t$

<u>$E(t) = E_0$</u>

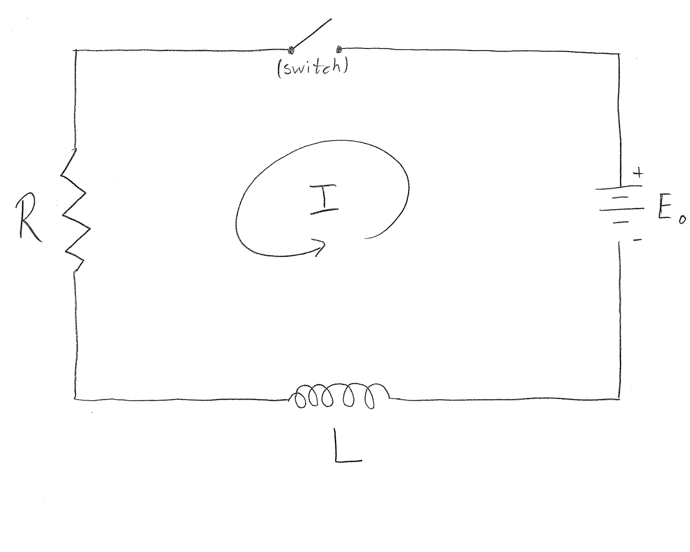
In the case where we have a battery attached, $E'(t) = 0$, $E(0) = E_0$, and we have $I'(0) = \frac{E_0}{L} - \frac{Q_0}{LC}$ by plugging $t = 0$ into (4.4) and solving for $I'(0)$ where Q_0 is the initial charge on the capacitor.

<u>$E(t) = E_0 \cos \omega t$</u>

In the case where we have an AC generator attached, $E'(t) = -\omega E_0 \sin \omega t$, $E(0) = E_0$, and we have $I'(0) = \frac{E_0}{L} - \frac{Q_0}{LC}$ by plugging $t = 0$ into (4.4) and solving for $I'(0)$ where Q_0 is the initial charge on the capacitor.

Exercises.

(1) A resistor and an inductor are connected in series with a battery of constant voltage E_0, as shown in Figure 1. The switch is closed at $t = 0$. Assuming $L = 0.2 henrys$, $R = 2 ohms$, and $E_0 = 4 volts$, find
   (a) a formula for the current as a function of t.
   (b) the voltage drop across the resistance and that across the inductance.



(2) In the circuit of Figure 2, assume that $L = 0.5 henrys$, $R = 2 ohms$, $C = 0.1 farads$, $E = 4 volts$, and $Q_0 = 0$. Find the current in the loop.

(3) In the circuit of Figure 2, assume that R, L, C, and Q_0 are the same as in Exercise 2 but that $E(t) = 2 \sin 4t$ volts. Find the current in the loop as a function of time.

4.7. Additional Exercises.

5. Laplace Transformations

5.1. The Laplace Transform.

In this chapter we will develop yet another method for solving differential equations with an initial value. The method we will use involves the *Laplace Transformation*. This transformation takes a function and gives another function. The difference between the two functions is that they live in "two different worlds"! Let's think of the original function being is the "standard" world, and the new function being in the "Laplace" world. We have the loose associations between the two worlds (by loose I mean up to a constant):

Standard	Laplace
Integration	Division by s
Differentiation	Multiplication by s

where s is as in the definition of the transform below. Surely this must seem surprising! However, while it is interesting, we will not explore that relation in this chapter...

5.1.1. *Definition of the Transform.*

Definition 1 (Laplace Transformation)**.** *Let $f(t)$ be a function defined on the interval $[0,\infty)$. Define $\mathcal{L}[f](s) = F(s)$ by the improper integral (when it is convergent):*

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

where s is a real number.

It is possible that there may be no s value for which the integral is convergent. We will only consider functions whose Laplace transform exists on intervals of the form (s_0, ∞) for some $s_0 \in \mathbb{R}$. Now you might ask "when can we guarantee that a function has a Laplace transform on some interval (a, ∞)?"

An answer to this question is if the function is of *exponential order*. This is not the only answer, but it is good enough for us right now.

Definition 2. *Let $f(t)$ and $g(t)$ be two functions. Suppose that there exist $M > 0$ and $N > 0$ such that*

$$|f(t)| \leq Mg(t)$$

whenever $t \geq N$. Then we say that $f(t)$ is of the order of $g(t)$. The notation for this statement is

$$f(t) = O[g(t)].$$

In particular, if $g(t) = e^{at}$ for some $a \in \mathbb{R}$, we say that $f(t)$ is of exponential order.

Theorem 1. *Let f be piecewise continuous on the interval $[0,\infty)$, and let $f(t) = O[e^{at}]$ for some $a \in \mathbb{R}$. Then the Laplace transform $F(s)$ exists at least for $s > a$.*

Proof. Since $f(t) = O[e^{at}]$ we have constants $M, N > 0$ such that $|f(t)| \leq Me^{at}$ for $t \geq N$. Then:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t)\, dt \\ &\leq \int_0^N e^{-st} f(t)\, dt + \int_N^\infty e^{-st} |f(t)|\, dt \\ &\leq \int_0^N e^{-st} f(t)\, dt + \int_N^\infty e^{-st} \left(Me^{at}\right)\, dt \\ &= \int_0^N e^{-st} f(t)\, dt + \int_N^\infty Me^{-(s-a)t}\, dt < \infty \end{aligned}$$

at least for $s > a$. □

5.1.2. *Examples of the Transform.*

Let's find the Laplace transform of a few simple functions:

Example 1. *Find the Laplace transform of the following functions:*

(a) $f(t) = c, \quad c \neq 0$
(b) $f(t) = t^n, \quad$ *n a positive integer*
(c) $f(t) = e^{at}, \quad a \in \mathbb{R}$
(d) $f(t) = \sin at, \quad a \neq 0$

Solution.

(a)

$$F(s)=\int_0^\infty ce^{-st}\,dt=c\lim_{R\to\infty}\int_0^R e^{-st}\,dt=c\lim_{R\to\infty}\left(-\frac{1}{s}e^{-st}\right)\Big|_0^R=\lim_{R\to\infty}\frac{-c}{s}e^{-sR}+\frac{c}{s}=\frac{c}{s},s>0.$$

(b) *We will make an induction argument on n to find this Laplace transform.*

($n=1$)

$$\begin{aligned}\int_0^\infty te^{-st}dt &\overset{IBP}{=} -\frac{t}{s}e^{-st}\Big|_0^\infty+\frac{1}{s}\int_0^\infty e^{-st}dt\\ &\overset{(a)}{=} 0+\frac{1}{s}\left(\frac{1}{s}\right)=\frac{1}{s^2}\end{aligned}$$

($n=2$)

$$\begin{aligned}\int_0^\infty t^2e^{-st}dt &\overset{IBP}{=} -\frac{t^2}{s}e^{-st}\Big|_0^\infty+\frac{2}{s}\int_0^\infty te^{-st}dt\\ &\overset{(n=1)}{=} 0+\frac{2}{s}\left(\frac{1}{s^2}\right)=\frac{2}{s^3}\end{aligned}$$

Inductive Hypothesis

$$\mathcal{L}[t^n]=\int_0^\infty t^ne^{-st}dt=\frac{n!}{s^{n+1}}$$

($n+1$ case)

$$\begin{aligned}\mathcal{L}[t^{n+1}] &= \int_0^\infty t^{n+1}e^{-st}dt\overset{IBP}{=}-\frac{t^{n+1}}{s}e^{-st}\Big|_0^\infty+\frac{n+1}{s}\int_0^\infty t^ne^{-st}dt\\ &\overset{(Ind.\ Hyp.)}{=} 0+\frac{n+1}{s}\left(\frac{n!}{s^{n+1}}\right)=\frac{(n+1)!}{s^{(n+1)+1}}\end{aligned}$$

So by induction we have that

$$F(s)=\frac{n!}{s^{n+1}},s>0.$$

(c)

$$F(s)=\int_0^\infty e^{-st}e^{at}dt=\int_0^\infty e^{-(s-a)t}dt=-\frac{1}{s-a}e^{-(s-a)t}\Big|_0^\infty$$

if we have that $s>a$ then the integral converges and is:

$$F(s)=\frac{1}{s-a},s>a.$$

(d)

$$\begin{aligned}F(s) &= \int_0^\infty e^{-st}\sin at\,dt\overset{IBP}{=}-\frac{e^{-st}}{a}\cos at\Big|_0^\infty-\frac{s}{a}\int_0^\infty e^{-st}\cos at\,dt\\ &= \frac{1}{a}-\frac{s}{a}\int_0^\infty e^{-st}\cos at\,dt\overset{IBP}{=}\frac{1}{a}-\frac{s}{a}\left(\frac{e^{-st}}{a}\sin at\Big|_0^\infty+\frac{s}{a}\int_0^\infty e^{-st}\sin at\,dt\right)\\ &= \frac{1}{a}-\frac{s}{a}\left(\frac{s}{a}\int_0^\infty e^{-st}\sin at\,dt\right)=\frac{1}{a}-\frac{s^2}{a^2}\int_0^\infty e^{-st}\sin at\,dt\end{aligned}$$

Isolating the integral with sine we get:

$$\left(1+\frac{s^2}{a^2}\right)\int_0^\infty e^{-st}\sin at\,dt=\frac{1}{a}$$

which gives:

$$\mathcal{L}[\sin at]=\frac{1}{a}\left(\frac{1}{1+\frac{s^2}{a^2}}\right)=\frac{1}{a}\left(\frac{1}{\frac{s^2+a^2}{a^2}}\right)=\frac{a}{s^2+a^2}.$$

More compactly:

$$F(s)=\frac{a}{s^2+a^2},s>0.$$

◇

Exercises. Find the Laplace transform of the given function using the definition of the transform.

(1) $f(t) = \cos at$

(2) $f(t) = te^{at}$

(3) $f(t) = t \sin at$

(4) $f(t) = t^n e^{at}$

Recall that $\cosh bt = \dfrac{e^{bt} + e^{-bt}}{2}$ and $\sinh bt = \dfrac{e^{bt} - e^{-bt}}{2}$. Using this, find the Laplace transform of the given functions.

(5) $f(t) = \cosh bt$

(6) $f(t) = \sinh bt$

(7) $f(t) = e^{at} \cosh bt$

(8) $f(t) = e^{at} \sinh bt$

Recall that $\cos bt = \dfrac{e^{ibt} + e^{-ibt}}{2}$ and $\sin bt = \dfrac{e^{ibt} - e^{-ibt}}{2i}$. Using this, and assuming that the necessary integration formulas extend to this case, find the Laplace transform of the given functions.

(9) $f(t) = \cos bt$

(10) $f(t) = \sin bt$

(11) $f(t) = e^{at} \cos bt$

(12) $f(t) = e^{at} \sin bt$

Challenge Problems:

(13) The gamma function is defined as:

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx.$$

(a) Show that, for $n > 0$,

$$\Gamma(n+1) = n\Gamma(n).$$

(b) Show that $\Gamma(1) = 1$.

(c) If n is a positive integer, show that

$$\Gamma(n+1) = n!.$$

(d) Show that, for $n > 0$,

$$n(n+1)(n+2)\cdots(n+k-1) = \frac{\Gamma(n+k)}{\Gamma(n)}.$$

(14) Consider the Laplace transform of t^p, where $p > -1$.

(a) Referring to the previous exercise, show that

$$\mathcal{L}[t^p] = \int_0^\infty e^{-st} t^p dt = \frac{1}{s^{p+1}} \int_0^\infty x^p e^{-x} dx = \frac{\Gamma(p+1)}{s^{p+1}}, s > 0.$$

(b) Show that

$$\mathcal{L}[t^{-\frac{1}{2}}] = \frac{2}{\sqrt{s}} \int_0^\infty e^{-x^2} dx, s > 0.$$

It is possible to show that

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2};$$

hence

$$\mathcal{L}[t^{-\frac{1}{2}}] = \sqrt{\frac{pi}{s}}, s > 0.$$

(c) Show that

$$\mathcal{L}[t^{\frac{1}{2}}] = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}, s > 0.$$

5.2. Properties of the Laplace Transform.

In this section we will give various properties of the Laplace transform as well as a table of common Laplace transformations.

Theorem 1 (Properties of the Laplace Transform)**.** *Suppose f and g are piecewise continuous on the interval $[0,\infty)$ and that $f(t)=O[e^{at}]$ and $g(t)=O[e^{bt}]$, then $F(s)$ exists for $s>a$ and $G(s)$ exists for $s>b$. Then the Laplace transform has the following properties:*

(a) *(Linearity)*

$$\mathcal{L}[c_1 f(t)+c_2 g(t)] = c_1F(s)+c_2G(s)$$

for $s>\max\{a,b\}$.

(b) *(The Shifting Theorem)*
If $h(t)=e^{ct}f(t)$, then $H(s)=F(s-c)$ for $s>a+c$.

(c) *If $k(t)=\displaystyle\int_0^t f(u)\,du$, then $K(s)=\dfrac{1}{s}F(s)$ for $s>\max\{a,0\}$.*

(d) *If $p_n(t)=t^nf(t)$, then $P_n(t)=(-1)^n\dfrac{d^nF(s)}{ds^n}$, for $s>a$.*

(e) *If $q(t)=\begin{cases} 0, & 0<t<c \\ f(t-c), & t>c\end{cases}$, then $Q(s)=e^{-cs}F(s)$, $s>a$.*

(f) *Suppose that $f^{(n-1)}=O[e^{at}]$, $f,f',\dots,f^{(n-1)}$ are continuous on $[0,\infty)$, and that $f^{(n)}$ is piecewise continuous on $[0,\infty)$. Then $\mathcal{L}[f^{(n)}](s)$ exists for $s>\max\{a,0\}$ and*

$$\mathcal{L}[f^{(n)}] = s^nF(s)-\left[s^{n-1}f(0)+s^{n-1}f'(0)+\cdots+sf^{(n-2)}(0)+f^{(n-1)}(0)\right].$$

Proof.

(a)

$$\mathcal{L}[c_1f(t)+c_2g(t)] = \int_0^\infty e^{-st}(c_1f(t)+c_2g(t))\,dt = \int_0^\infty e^{-st}c_1f(t)\,dt+\int_0^\infty e^{-st}c_2g(t)\,dt = c_1F(s)+c_2G(s).$$

(b)

$$H(s)=\int_0^\infty e^{-st}e^{ct}f(t)\,dt=\int_0^\infty e^{-(s-c)t}f(t)\,dt = F(s-c)$$

(c)

$$K(s)=\int_0^\infty\left(e^{-st}\int_0^t f(u)\,du\right)dt$$

To evaluate this integral we need to use integration by parts with the following choices of v and dw:

$v=\int_0^t f(u)\,du$	$dw=e^{-st}dt$
$dv=f(t)\,dt$	$w=-\frac{1}{s}e^{-st}$

Then we can continue the integral:

$$K(s)=\int_0^\infty\left(e^{-st}\int_0^t f(u)\,du\right)dt=\left(-\frac{1}{s}e^{-st}\int_0^t f(u)\,du\right)\Big|_0^\infty+\frac{1}{s}\int_0^\infty e^{-st}f(t)\,dt = 0+\frac{1}{s}F(s)=\frac{1}{s}F(s).$$

□

Since part (f) of this theorem will be particularly helpful to us in solving differential equations, let's do a few examples of it, but first let's write down the general forms of $\mathcal{L}[f']$ and $\mathcal{L}[f'']$ (assuming that $\mathcal{L}[f]=F$) :

$$\mathcal{L}[f'] = sF(s)-f(0)$$
$$\mathcal{L}[f''] = s^2F(s)-sf(0)-f'(0)$$

Example 1. *Assuming that $\mathcal{L}[f]=F$, find the Laplace transform of the indicated derivative of f in terms of F:*

(a) *f', if $f(0)=3$.*
(b) *f'', if $f(0)=1$ and $f'(0)=2$.*

Solution.

(a) *Using the formulas above:*

$$\mathcal{L}\left[f'\right] = sF(s)-f(0) = sF(s)-3.$$

(b) *Again using the formulas above:*

$$\mathcal{L}[f''] = s^2F(s) - sf(0) - f'(0) = s^2F(s) - s - 2.$$

◇

Property (Table of Common Laplace Transforms)**.**

$f(t)$	$F(s)$
c	$\frac{c}{s}$, $s > 0$
t^n	$\frac{n!}{s^{n+1}}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
$t^n e^{at}$ *(n a positive integer)*	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$t^p e^{at}$ $(p > -1)$	$\frac{\Gamma(p+1)}{s^{p+1}}$, $s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$, $s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$, $s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$, $s > \lvert a\rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$, $s > \lvert a\rvert$
$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$, $s > 0$
$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$, $s > 0$
$t\sinh at$	$\frac{2as}{(s^2-a^2)^2}$, $s > \lvert a\rvert$
$t\cosh at$	$\frac{s^2+a^2}{(s^2-a^2)^2}$, $s > \lvert a\rvert$
$\sin at - at\cos at$	$\frac{2a^3}{(s^2+a^2)^2}$, $s > 0$
$at\cosh at - \sinh at$	$\frac{2a^3}{(s^2-a^2)^2}$, $s > \lvert a\rvert$
$e^{ct}f(t)$	$F(s-c)$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$, $c > 0$
$f^{(n)}(t)$	$s^nF(s) - [s^{n-1}f(0) + s^{n-2}f'(0) + \cdots + f^{(n-1)}(0)]$
$(-t)^n f(t)$	$F^{(n)}(s)$

Using the table above, combined with Theorem 1, we can easily compute the Laplace transformations of several functions. Here are some examples:

Example 2. *Compute the Laplace transform of the following functions:*

(a) $f(t) = e^{-2t}\cos 3t$

(b) $f(t) = t^2 \sin t$

(c) $f(t) = \begin{cases} 0, & 0 < t < 1 \\ (t-1)^2, & t > 1 \end{cases}$

Solution.

(a) *If we let* $g(t) = \cos 3t$*, then* $f(t) = e^{-2t} g(t)$ *so we can use Theorem 1.b to get that (with* $c = -2$*):*

$$F(s) = G(s-c) = \frac{s+2}{(s+2)^2+9}$$

since $G(s) = \dfrac{s}{s^2+9}$ *from the table above.*

(b) *If we let* $g(t) = \sin t$*, then* $G(s) = \dfrac{1}{s^2+1}$ *and* $f(t) = t^2 g(t)$*, so by Theorem 1.d we get (with* $n = 2$*):*

$$F(s) = (-1)^2 \frac{d^2G}{ds^2} = \frac{d^2}{ds^2}\left[\frac{1}{s^2+1}\right] = \frac{6s^2-2}{(s^2+1)^3}.$$

(c) *If we let* $g(t) = t^2$*, then* $G(s) = \dfrac{2}{s^3}$ *and by Theorem 1.e we have (with* $c = 1$*):*

$$F(s) = e^{-cs} G(s) = e^{-s}\frac{2}{s^3} = \frac{2}{s^3}e^{-s}.$$

◇

Exercises. Find the Laplace transform of the following functions:

(1) $f(t) = 2e^{-t} - 3\sin 4t$

(2) $f(t) = \cosh 2t$

(3) $f(t) = e^{2t}\sin 3t$

(4) $f(t) = e^{-t}\cos 2t$

(5) $f(t) = e^{-3t}t^4$

(6) $f(t) = t^3 e^{4t}$

(7) $f(t) = t^2\cos t$

(8) $f(t) = t\sin 2t$

(9) $g(t) = e^{2t}\sqrt{t}$

(10) $g(t) = \int_0^t \sin 2u\, du$

(11) $g(t) = \int_0^t x^2 e^x\, dx$

(12) $g(t) = \int_0^t \cos^2 u\, du$

(13) $g(t) = \begin{cases} 0, & 0 < t < 2 \\ 1, & t > 2 \end{cases}$

(14) $g(t) = \begin{cases} 0, & 0 < t < \pi \\ \sin(t-\pi), & t > \pi \end{cases}$

(15) $g(t) = \begin{cases} 0, & 0 < t < 1 \\ t^2, & t > 1 \end{cases}$

(16) $g(t) = \begin{cases} 0, & 0 < t < 1 \\ (t-1)e^t, & t > 1 \end{cases}$

Let $\mathcal{L}[f] = F$. Find the Laplace transform of the given derivative of f:

(17) f'', if $f(0) = -3$ and $f'(0) = 0$

(18) f''', if $f(0) = 1$, $f'(0) = 0$, and $f''(0) = -5$

(19) f'', if $f(0) = -4$ and $f'(0) = -9$

(20) f'', if $f(0) = 8$ and $f'(0) = 33$

(21) f'', if $f(0) = 56$ and $f'(0) = 19$

(22) f'', if $f(0) = 23$ and $f'(0) = -11$

Challenge Problems:

(23) Prove Theorem 1.d.

(24) Prove Theorem 1.e.

(25) If $g(t) = f(ct)$, where c is a positive constant, show that:

$$G(s) = \frac{1}{c} F\left(\frac{s}{c}\right).$$

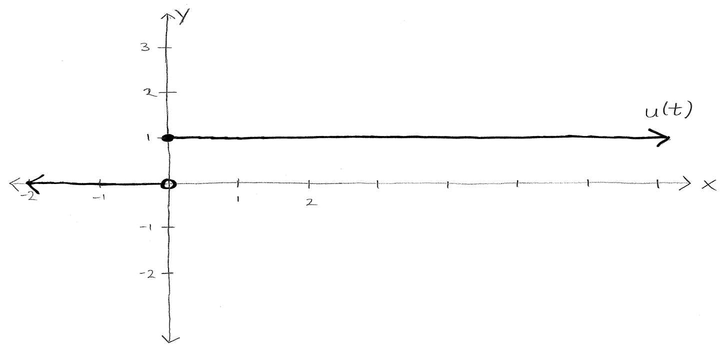
5.3. Some Special Functions.

5.3.1. *Step Functions.*

Recall the unit step function $u(t)$ given by

$$u(t) := \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

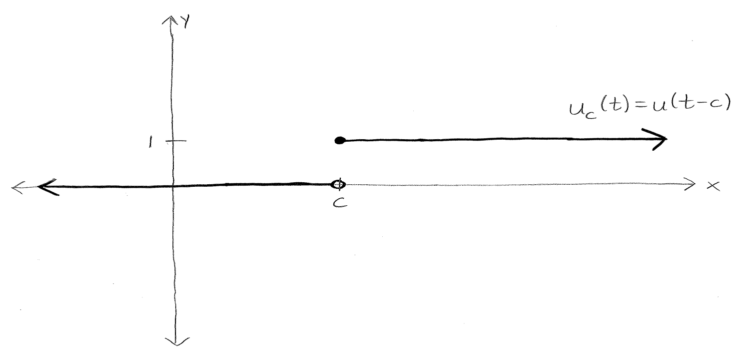
The graph of this function is:



We will now define a variant of this function that, instead of stepping up to 1 at $t = 0$, will step up to 1 at time $t = c$. Define the function $u_c(t)$ as follows:

$$u_c(t) := u(t-c) = \begin{cases} 0, & x < c \\ 1, & x \geq c \end{cases}$$

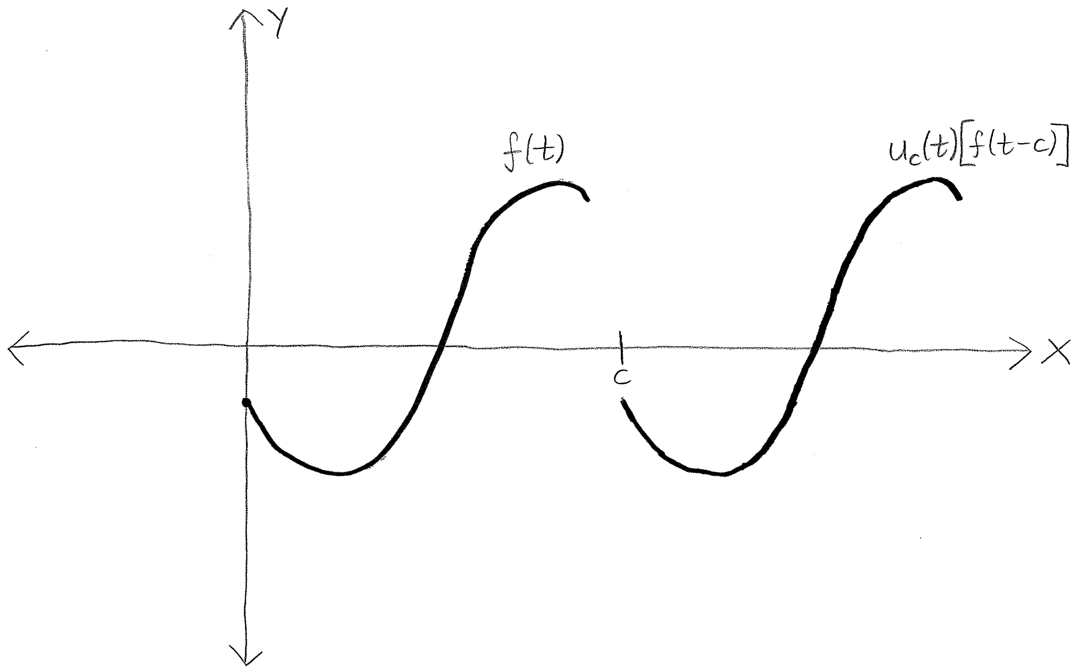
The graph of this function is:



We will be primarily concerned with the case in which c is positive since we are only going to be interested in cases in which $t > 0$. You can think of the function $u_c(t)$ as being a switch that you turn on at time $t = c$. If we multiply $u_c(t)$ onto any function $f(t)$ as follows, we can think of turning on the function $f(t)$ at time $t = c$. The way to get a function to start at time $t = c$ is to write it as follows:

$$u_c(t)[f(t-c)] := \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases}$$

Observe the picture below:



See how $u_c(t)[f(t-c)]$ is the exact same as $f(t)$ except it starts at time $t = c$ rather than time $t = 0$? This is especially useful in applications such as those in Section 4.6 (think of turning the switch on and off and maybe changing the input voltage function while the switch is open).

We would now like to take the Laplace to take the Laplace transformation of $u_c(t)[f(t-c)]$. Notice that by Theorem 1.e of Section 5.2 we have that:

$$\mathcal{L}\left[u_c(t)[f(t-c)]\right] = e^{-cs}F(s),$$

where $F(s) = \mathcal{L}[f](s)$.

We have two useful formulas. These will allow us to go back and forth between piecewise notation and step function notation:

$$f(t) = \begin{cases} f_1(t), & 0 < t < c \\ f_2(t), & t \geq c \end{cases} \quad \leftrightarrow \quad f(t) = f_1(t) + u_c(t)[f_2(t) - f_1(t)]$$

and

$$f(t) = f_1(t) + u_c(t)[f_2(t)] \quad \leftrightarrow \quad f(t) = \begin{cases} f_1(t), & 0 < t < c \\ f_1(t) + f_2(t), & t \geq c \end{cases}$$

Example 1. *Switch from piecewise notation to step function notation:*

(a)

$$f(t) = \begin{cases} t^2, & 0 < t < 3 \\ \sin 2t, & t \geq 3 \end{cases}$$

(b)

$$f(t) = \begin{cases} 0, & 0 < t < 5 \\ \ln t, & t \geq 5 \end{cases}$$

Solution.

(a)

$$f(t) = t^2 + u_3(t)[\sin 2t - t^2]$$

(b)

$$f(t) = u_5(t)[\ln t]$$

◇

Example 2. *Switch from step function notation to piecewise notation:*

(a)

$$f(t) = u_1(t)[t-1] - 2u_2(t)[t-2] + u_3(t)[t-3]$$

(b)

$$f(t) = \cos t^2 + u_3(t)[t]$$

Solution.

(a) *First let's rewrite the function*

$$f(t) = u_1(t)[t-1] + u_2(t)[-2(t-2)] + u_3(t)[t-3].$$

Then we have

$$f(t) = \begin{cases} 0, & 0 < t < 1 \\ t-1, & 1 \le t < 2 \\ t-1-2(t-2), & 2 \le t < 3 \\ t-1-2(t-2)+t-3, & 3 \le t \end{cases}$$

or simplified:

$$f(t) = \begin{cases} 0, & 0 < t < 1 \\ t-1, & 1 \le t < 2 \\ 3-t, & 1 \le t < 3 \\ 0, & 3 \le t \end{cases}$$

(b)

$$f(t) = \begin{cases} \cos t^2, & 0 < t < 3 \\ \cos t^2 + t, & t \ge 3 \end{cases}$$

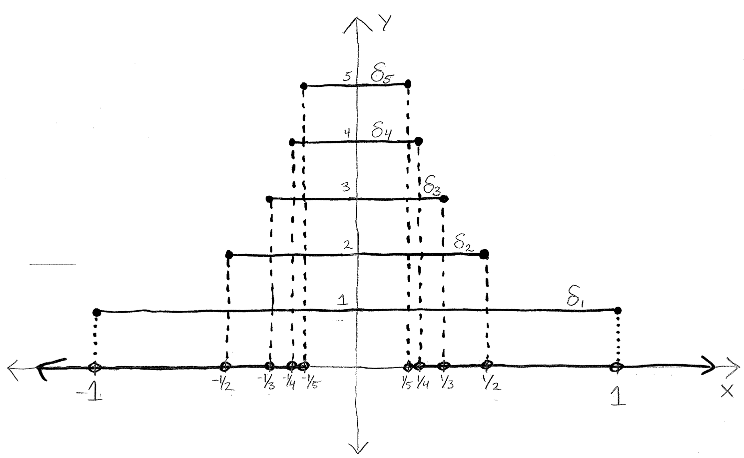
◇

5.3.2. *Dirac-Delta Distribution.*

Let's define a sequence of functions as follows:

$$\delta_n(t) = \begin{cases} 0, & t < -\frac{1}{n} \\ n, & -\frac{1}{n} \le t \le \frac{1}{n} \\ 0, & t > \frac{1}{n} \end{cases}$$

for $n = 1, 2, 3, \ldots$. Below are the graphs of the first five functions in the sequence: δ_1, δ_2, δ_3, δ_4, and δ_5:



These functions converge to a distribution (or generalized function) known as the *Dirac-Delta Distribution* $\delta(t)$ given by:

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

One interesting feature of the Dirac-Delta distribution is the following:

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1,$$

and maybe even more interesting is:

$$\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0).$$

Now we can also define the *Shifted Dirac-Delta distribution* as follows:

$$\delta_{t_0}(t) = \delta(t - t_0)$$

which essentially looks like:

$$\delta(t) = \begin{cases} 0, & t \neq t_0 \\ \infty, & t = t_0 \end{cases}$$

Just as the Dirac-Delta distribution has interesting properties when integrated, so does its shifted counterpart:

$$\int_{-\infty}^{\infty} \delta_{t_0}(t)\,dt = 1,$$

and

$$\int_{-\infty}^{\infty} f(t)\delta_{t_0}(t)\,dt = f(t_0). \tag{5.1}$$

Using equation (5.1), we will now take the Laplace transform of $\delta_{t_0}(t)$ (where we assume that $t_0 \geq 0$):

$$\mathcal{L}\left[\delta_{t_0}(t)\right] = \int_0^{\infty} e^{-st}\delta_{t_0}(t)\,dt = e^{-st_0}$$

by Equation (5.1) where we let $f(t) = e^{-st}$.

5.3.3. *Periodic Functions.*

Exercises. Find the Laplace transform of the given function:

(1) $f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$

(2) $g(t) = u_\pi(t)[t^2]$

(3) $h(t) = t - u_1(t)[t-1]$

(4) $f(t) = \begin{cases} 0, & t < \pi \\ t - \pi, & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi \end{cases}$

(5) $g(t) = u_\pi(t)\,[\cos t]$

Challenge Problems:

(6) Assume $k > 0$. Show

$$\mathcal{L}^{-1}[F(ks)] = \frac{1}{k} f\left(\frac{t}{k}\right).$$

(7) Let $a, b \in \mathbb{R}$ with $a > 0$. Show

$$\mathcal{L}^{-1}[F(as+b)] = \frac{1}{a} e^{-\frac{bt}{a}} f\left(\frac{t}{a}\right).$$

(8) Use Exercise 7 to find the inverse Laplace transform of:

(a) $F(s) = \dfrac{2^{n+1}n!}{s^{n+1}}$

(b) $F(s) = \dfrac{2s+1}{4s^2+4s+5}$

(c) $F(s) = \dfrac{1}{9s^2-12s+3}$

(d) $G(s) = \frac{e^2 e^{-4s}}{2s-1}$

5.4. The Inverse Laplace Transform.

Now that we have had plenty of practice going from the "normal" world to the world of "Laplace", we should learn how to go back. For the purposes of these notes, we will not need any high powered theorems; we will just be doing the opposite of what we did in Section 5.2. So instead of starting on the left side of the table in 5.2, we will try to get the function we are trying to take the inverse transform of and try to get it into a sum of forms on the right side of the table so we can just use the linearity of the Laplace transform to go back. This may seem unclear, but it should become clear after a few examples. To do this process, you will have to be VERY comfortable with partial fractions as well as completing the square.

Example 1. *Find the inverse Laplace transform of the following functions:*

(a)

$$F(s) = \frac{3}{s^2+4}$$

(b)

$$G(s) = \frac{2s-3}{s^2+2s+10}$$

(c)

$$H(s) = \frac{4}{(s-1)^3}$$

Solution.

(a) *This almost looks like the transform of* $\sin 2t$*, but the top should be a* 2*, not a* 3*; however this is easily cured:*

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{3}{s^2+4}\right] = \mathcal{L}^{-1}\left[\frac{3}{2}\frac{2}{s^2+4}\right] = \frac{3}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] = \frac{3}{2}\sin 2t.$$

(b) *To begin with, for this one, we need to complete the square*

$$G(s) = \frac{2s-3}{s^2+2s+10} = \frac{2s-3}{(s^2+2s+1)+9} = \frac{2s-3}{(s+1)^2+9} = \frac{2s}{(s+1)^2+9} - \frac{3}{(s+1)^2+9}.$$

Now we will have to play some tricks again:

$$\begin{aligned} G(s) &= \frac{2s}{(s+1)^2+9} - \frac{3}{(s+1)^2+9} = \frac{2(s+1-1)}{(s+1)^2+9} - \frac{3}{(s+1)^2+9} \\ &= \frac{2(s+1)-2}{(s+1)^2+9} - \frac{3}{(s+1)^2+9} = \frac{2(s+1)}{(s+1)^2+9} - \frac{5}{(s+1)^2+9} \\ &= 2\frac{s+1}{(s+1)^2+9} - \frac{5}{3}\frac{3}{(s+1)^2+9} \end{aligned}$$

which is the Laplace transform of

$$g(t) = 2e^{-t}\cos 3t - \frac{5}{3}e^{-t}\sin 3t.$$

(c) *This one is going to be a bit trickier. If we let* $a=1$ *and* $n=2$ *then* $H(s)$ *looks an awful lot like the Laplace transform of* t^2e^t *which is* $\frac{2!}{(s-1)^{2+1}} = \frac{4}{(s-1)^3}$ *because it is. Thus*

$$\mathcal{L}^{-1}[H(s)] = t^2e^t.$$

◇

Example 2. *Find the inverse Laplace transform of:*

$$F(s) = \frac{2e^{-2s}}{s^2-4}.$$

Solution. *This is the transform of* $f(t) = u_2(t)[\sinh 2(t-2)]$.

◇

Exercises. Find the inverse Laplace transform of:

(1) $G(s) = \dfrac{a_1}{s} + \dfrac{a_2}{s^2} + \dfrac{a_3}{s^3}$

(2) $F(s) = \dfrac{1}{s^2 + 9}$

(3) $G(s) = \dfrac{4(s+1)}{s^2 - 16}$

(4) $H(s) = \dfrac{1}{(s+1)(s+2)}$

(5) $F(s) = \dfrac{s+3}{(s+1)^2 + 1}$

(6) $J(s) = \dfrac{1}{(s-2)^3} + \dfrac{1}{(s-2)^5}$

(7) $K(s) = \dfrac{e^{-3s} - e^{-s}}{s}$

(8) $F(s) = \dfrac{e^{-s} + e^{-2s} - 3e^{-3s} + e^{-6s}}{s^2}$

(9) $Q(s) = \dfrac{2s+1}{s^2 + 4s + 13}$

(10) $G(s) = \dfrac{e^{-\pi s}}{s^2 + 2s + 2}$

(11) $F(s) = \dfrac{s}{(s^2 + a^2)(s^2 + b^2)}$

5.5. Convolution.

In this section we will be primarily concerned with taking the inverse Laplace transform of a product of known Laplace transforms. To say it symbolically, we will be finding

$$\mathcal{L}^{-1}[F(s)G(s)]$$

where $F(s)$ and $G(s)$ are the Laplace transforms of two known functions $f(t)$ and $g(t)$ respectively.

Definition 1 (Convolution)**.** *Let $f(t)$ and $g(t)$ be two functions. Then the function $(f*g)(t)$ defined by:*

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

is called the convolution of f and g.

The convolution operator satisfies several algebraic properties:

Theorem 1. *The convolution operator satisfies:*

(a) *(Commutativity)*

$$f*g = g*f$$

(b) *(Distributivity)*

$$f*(g_1+g_2) = f*g_1 + f*g_2$$

(c) *(Associativity)*

$$f*(g*h) = (f*g)*h$$

(d)

$$f*0 = 0*f = 0$$

Proof. Left to the reader. □

Theorem 2. *The inverse Laplace transform of $H(s) = F(s)G(s)$ where $F(s)$ and $G(s)$ are the Laplace transforms of two known functions $f(t)$ and $g(t)$ respectively is $(f*g)(t)$.*

Proof. Let's start by noting that:

$$F(s) = \int_0^\infty e^{-sr}f(r)\,dr$$

and

$$G(s) = \int_0^\infty e^{-s\tau}g(\tau)\,d\tau.$$

The variable of integration is different for a reason, however it does not have any effect on the integral other than to distinguish the two. Now we have:

$$F(s)G(s) = \int_0^\infty e^{-sr}f(r)\,dr \int_0^\infty e^{-s\tau}g(\tau)\,d\tau.$$

Then since the first integral does not depend on the variable of integration of the second we can write:

$$F(s)G(s) = \int_0^\infty g(\tau)\,d\tau \int_0^\infty e^{-s(r+\tau)}f(r)\,dr.$$

Notice that this is an iterated integral. Now let's use the change of variable $r = t-\tau$ for a fixed value of τ (so that $dt = d\tau$) to get:

$$F(s)G(s) = \int_0^\infty g(\tau)\,d\tau \int_0^\infty e^{-st}f(t-\tau)\,dt = \int_0^\infty g(\tau)\,d\tau \int_0^\infty e^{-st}f(t-\tau)\,dt.$$

Switching the order of integration we get:

$$F(s)G(s) = \int_0^\infty e^{-st}\int_0^t f(t-\tau)g(\tau)\,d\tau\,dt = \int_0^\infty e^{-st}(f*g)(t)\,dt.$$

Thus we have shown that $\mathcal{L}^{-1}[H(s)] = (f*g)(t)$. □

Example 1. *Find the convolution of the two functions:*

$$f(t) = 3t \quad \textit{and} \quad g(t) = \sin 5t.$$

Solution.

$$
\begin{aligned}
(f*g)(t) &= \int_0^t f(t-\tau)g(\tau)\,d\tau \\
&= \int_0^t 3(t-\tau)\sin 5\tau\,d\tau \\
&= 3\int_0^t (t-\tau)\sin 5\tau\,d\tau
\end{aligned}
$$

Now using integration by parts with

$u = t-\tau$	$dv = \sin 5\tau\,d\tau$
$du = -d\tau$	$v = -\frac{1}{5}\cos 5\tau$

$$
\begin{aligned}
(f*g)(t) &= 3\int_0^t (t-\tau)\sin 5\tau\,d\tau \\
&= -\frac{3}{5}(t-\tau)\cos 5\tau\Big|_0^t - \frac{3}{5}\int_0^t \cos 5\tau\,d\tau \\
&= 0 - \left(-\frac{3}{5}t\right) - \frac{3}{25}\sin 5\tau\Big|_0^t \\
&= \frac{3}{5}t - \frac{3}{25}\sin 5t
\end{aligned}
$$

◇

Now let's see how this is useful for finding inverse Laplace transforms:

Example 2. *Find the inverse Laplace transform of:*

$$F(s) = \frac{s}{(s+1)(s^2+4)}.$$

Solution. *Notice the following:*

$$F(s) = \frac{1}{s+1}\frac{s}{s^2+4}.$$

If we let $G(s) = \frac{1}{s+1}$ *and* $H(s) = \frac{s}{s^2+4}$. *Then by Theorem 2 (henceforth referred to as the Convolution Theorem) since* $F(s) = G(s)H(s)$ *we have*

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[G(s)H(s)] = \mathcal{L}^{-1}[G(s)] * \mathcal{L}^{-1}[H(s)] = e^{-t} * \cos 2t = \int_0^t e^{-(t-\tau)}\cos 2\tau\,d\tau.$$

◇

Exercises. Find the Laplace transform of the given function.

(1) $f(t) = \int_0^t (t-\tau)^2 \cos 2\tau\,d\tau$

(2) $g(t) = \int_0^t e^{-(t-\tau)}\sin\tau\,d\tau$

(3) $h(t) = \int_0^t (t-\tau)e^{\tau}\,d\tau$

(4) $k(t) = \int_0^t \sin(t-\tau)\cos\tau\,d\tau$

Find the inverse Laplace transform of the given function using the Convolution theorem.

(5) $F(s) = \frac{1}{s^4(s^2+1)}$

(6) $G(s) = \frac{s}{(s+1)(s^2+4)}$

(7) $H(s) = \frac{1}{(s+1)^2(s^2+4)}$

(8) $K(s) = \dfrac{Q(s)}{s^2+1}$ where $Q(s) = \mathcal{L}[q(t)]$

Challenge Problems:

(9) Prove Theorem 1 of this section.

(10) Find an example showing that $(f * 1)(t)$ need not be equal to $f(t)$.

(11) Show that $f * f$ need not be nonnegative. *Hint:* Use the example $f(t) = \sin t$.

5.6. **Application to Initial Value Problems.**

Finally getting back to differential equations, we will now apply the method of Laplace transformations to solving initial value problems which are linear and have constant coefficients. The only catch is that the initial values must all be at $t = 0$ due to the definition of the Laplace transformation.

5.6.1. 1^{st} *Order IVPs.*

Let's begin by solving equations of the form:

$$ay' + by = f(t); y(0) = c$$

where y is a function of t.

Remember that we have:

$$\mathcal{L}[y'](s) = sY(s) - y(0),$$

so since the Laplace transform is linear, if we take the Laplace transform of both sides of the equation we get:

$$a\left(sY(s) - y(0)\right) + bY(s) = F(s)$$

where $F(s) = \mathcal{L}[f(t)]$. The idea is to solve for $Y(s)$ and then take the inverse Laplace transform to find $y(t)$. The $y(t)$ you found will be the solution to the IVP. Let's see an example of this:

Example 1. *Solve the IVP:*

$$y' + 2y = e^{3t}; \quad y(0) = 1.$$

Solution. *Let's begin by taking the Laplace transform of the differential equation:*

$$sY(s) - y(0) + 2Y(s) = \frac{1}{s-3}.$$

Now plug in the initial value:

$$sY(s) - 1 + 2Y(s) = \frac{1}{s-3}.$$

Now solve for $Y(s)$:

$$(s+2)Y(s) = \frac{1}{s-3} + 1$$

which gives

$$Y(s) = \frac{1}{(s-3)(s+2)} + \frac{1}{s+2}$$

and using partial fractions:

$$Y(s) = \frac{1}{5}\frac{1}{s-3} - \frac{1}{5}\frac{1}{s+2} + \frac{1}{s+2} = \frac{1}{5}\frac{1}{s-3} + \frac{4}{5}\frac{1}{s+2}.$$

Now we can take the inverse Laplace transform and get:

$$y(t) = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}$$

as our final answer.

◇

5.6.2. 2^{nd} *Order IVPs.*

Now we will focus on solving equations of the form:

$$ay'' + by' + cy = f(t); \quad y(0) = k_1, y'(0) = k_2.$$

We already know that

$$\mathcal{L}[y'] = sY(s) - y(0),$$

now remember that

$$\mathcal{L}[y''] = s^2Y(s) - sy(0) - y'(0).$$

Using the above two statements we can solve second order IVPs. If we take the Laplace transformation of both sides of the second order equation above we have:

$$a\left(s^2Y(s) - sy(0) - y'(0)\right) + b\left(sY(s) - y(0)\right) + cY(s) = F(s).$$

Then, just as in the first order case, we solve for $Y(s)$ and take the inverse Laplace transform to find our desired solution: $y(t)$. Let's see some examples of this:

Example 2. *Solve the given IVPs:*

(a)

$$y'' + 3y' + 2y = 6e^t; \quad y(0) = 2, y'(0) = -1$$

(b)
$$y'' + 4y = \sin t - u_\pi(t)\,[\sin t]\,; \quad y(0) = y'(0) = 0$$

(c)
$$y'' + 2y' + 2y = \delta_\pi(t); \quad y(0) = 1, y'(0) = 0$$

Solution.

(a) *Let's begin by taking the Laplace transform of both sides:*
$$s^2Y(s) - sy(0) - y'(0) + 3\,(sY(s) - y(0)) + 2Y(s) = \frac{6}{s-1}.$$
Now let's plug in the initial values:
$$s^2Y(s) - 2s + 1 + 3\,(sY(s) + 1) + 2Y(s) = \frac{6}{s-1},$$
now simplify:
$$s^2Y(s) - 2s + 1 + 3sY(s) + 3 + 2Y(s) = \frac{6}{s-1}.$$
Next we solve for $Y(s)$:
$$\begin{aligned} s^2Y(s) + 3sY(s) + 2Y(s) - 2s + 4 &= \frac{6}{s-1} \\ (s^2 + 3s + 2)Y(s) &= 2s - 4 + \frac{6}{s-1} \\ Y(s) &= \frac{2s-4}{s^2+3s+2} + \frac{6}{s-1} \end{aligned}$$

Now we need to get $Y(s)$ into a form we can take the inverse Laplace transform of. Let's start by seeing if we can factor the denominator of the first term in $Y(s)$:
$$s^2 + 3s + 2 = (s+2)(s+1).$$
Thus we should use partial fractions to get this into a form which we can take the inverse transform of:
$$\frac{2s-4}{s^2+3s+2} = \frac{2s-4}{(s+2)(s+1)} = \frac{8}{s+2} - \frac{6}{s+1}$$
thus:
$$Y(s) = \frac{8}{s+2} - \frac{6}{s+1} + \frac{6}{s-1}$$
which has the inverse transform:
$$y(t) = 8e^{-2t} - 6e^{-t} + 6e^{t}.$$

(b) *Just as above, we will begin by taking the Laplace transform of the differential equation... Except that there is one small problem! How do we take the Laplace transform of $f(t) = \sin t - u_\pi(t)\,[\sin t]$? The first $\sin t$ isn't a problem, but the $u_\pi(t)\,[\sin t]$ is! We need to get this in the proper form to take the transform of, this means that instead of a "t", we need a "$t-\pi$". So let's fix this by adding zero:*
$$\sin t = \sin(t - \pi + \pi) = \sin[(t-\pi) + \pi] = \sin(t-\pi)\cos\pi + \cos(t-\pi)\sin\pi = -\sin(t-\pi).$$
Thus
$$\sin t - u_\pi(t)\,[\sin t] = \sin t + u_\pi(t)\,[\sin(t-\pi)]$$
which has the Laplace transform:
$$\frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1}.$$
So the Laplace transform of the differential equation is:
$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1}.$$
Plugging in the initial values:
$$s^2Y(s) + 4Y(s) = \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1}.$$
Isolating $Y(s)$ we get:
$$Y(s) = \frac{1}{(s^2+1)(s^2+4)} + e^{-\pi s}\frac{1}{(s^2+1)(s^2+4)}.$$

Using partial fractions on $\dfrac{1}{(s^2+1)(s^2+4)}$ *we get:*

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{1}{s^2+4}.$$

Thus

$$Y(s) = \frac{1}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{1}{s^2+4} + e^{-\pi s}\frac{1}{3}\frac{1}{s^2+1} - e^{-\pi s}\frac{1}{3}\frac{1}{s^2+4}.$$

Getting this into a form we can take the inverse transform of:

$$Y(s) = \frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4} + e^{-\pi s}\frac{1}{3}\frac{1}{s^2+1} - e^{-\pi s}\frac{1}{6}\frac{2}{s^2+4}.$$

Thus:

$$y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t + u_\pi(t)\left[\frac{1}{3}\sin(t-\pi)\right] - u_\pi(t)\left[\frac{1}{6}\sin 2(t-\pi)\right],$$

which could be simplified to:

$$y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t + \frac{1}{6}u_\pi(t)\left[2\sin(t-\pi) - \sin 2(t-\pi)\right].$$

(c) *This one we should have no problem taking the Laplace transform of:*

$$s^2Y(s) - sy(0) - y'(0) + 2\left(sY(s) - y(0)\right) + 2Y(s) = e^{-\pi s},$$

and plugging in the initial values gives:

$$s^2Y(s) - s + 2\left(sY(s) - 1\right) + 2Y(s) = e^{-\pi s}.$$

Simplifying:

$$s^2Y(s) - s + 2sY(s) - 2 + 2Y(s) = (s^2+2s+2)Y(s) - (s+2) = e^{-\pi s}.$$

Solving for $Y(s)$ *we get:*

$$Y(s) = \frac{s+2}{s^2+2s+2} + e^{-\pi s}\frac{1}{s^2+2s+2}.$$

To get this in a form more acceptable for an inverse Laplace transform we will need to complete the square:

$$\begin{aligned} Y(s) &= \frac{s+2}{(s^2+2s+1)+1} + e^{-\pi s}\frac{1}{(s^2+2s+1)+1} \\ &= \frac{s+2}{(s+1)^2+1} + e^{-\pi s}\frac{1}{(s+1)^2+1} \\ &= \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1} + e^{-\pi s}\frac{1}{(s+1)^2+1} \end{aligned}$$

Thus our solution is:

$$y(t) = e^{-t}\cos t + e^{-t}\sin t + u_\pi(t)\left[e^{-(t-\pi)}\sin(t-\pi)\right].$$

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Exercises. Solve the given IVP:

(1) $y' + 2y = u_1(t) - u_5(t)$; $y(0) = 1$

(2) $y'' + 3y' + 2y = 1 - u_2(t)$; $y(0) = y'(0) = 0$

(3) $y' + y = t$; $y(0) = 0$

(4) $y' + 2y = \sin \pi t$; $y(0) = 0$

(5) $y'' - y = 2\sin t$; $y(0) = 2, y'(0) = 1$

(6) $y'' + 2y' + y = 4\sin t$; $y(0) = -2$, $y'(0) = 1$

(7) $y'' + 4y' + 5y = 25t$; $y(0) = -5, y'(0) = 7$

(8) $y'' + 4y = \delta_\pi(t) - \delta_{2\pi}(t)$; $y(0) = y'(0) = 0$

(9) $y'' + 2y' + 3y = \sin t + \delta_{3\pi}(t)$; $y(0) = y'(0) = 0$

(10) $y'' + y = u_{\frac{\pi}{2}}(t) + 3\delta_{\frac{3\pi}{2}}(t)$; $y(0) = y'(0) = 0$

(11) $y'' + y = \delta_{2\pi}(t) \cos t$; $y(0) = 0, y'(0) = 1$

5.7. **Volterra Equations.**

Exercises.

5.8. Additional Exercises.

6. Series Solutions to Differential Equations

6.1. Taylor Series.

Exercises.

6.2. **Ordinary Points.**

Exercises.

6.3. **Singular Points.**

Exercises.

6.4. Additional Exercises.

7. Systems of First-Order Linear Differential Equations

7.1. Eigenvalues and Eigenvectors of a 2×2 Matrix.

Exercises.

7.2. 2×2 Systems of First-Order Linear Differential Equations.

Now we will work on systems of first order linear equations of the form:

$$\mathbf{v}'(t) = A\mathbf{v}(t) \tag{7.1}$$

where $\mathbf{v}(\mathbf{t}) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which may be written in the form:

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \tag{7.2}$$

where we have dropped the (t) for simplicity.

We will begin this section by looking at a direct, calculus only, way of solving the system; then we will turn to linear algebra to make life easier.

7.2.1. *Calculus Approach.* Let's look at how to solve this system of equation using calculus techniques alone. Start by solving the first equation in (7.2) for y:

$$y = \frac{1}{b}\left(x' - ax\right).$$

Now take this and plug it into the second equation to get:

$$\frac{1}{b}\left(x'' - ax'\right) = cx + \frac{d}{b}\left(x' - ax\right).$$

Simplify this equation to arrive at:

$$x'' - (a+d)x' + (ad - bc)x = 0.$$

We can solve this equation for x using the methods of Section 2.1. Assume that we have found the general solution of this equation $x(t)$. Plug this into the first equation in (7.2) and simply solve for y. This is all that is needed to solve this system of equations!

Let's do an example or two:

Example 1. *Solve the system of equation:*

$$\begin{cases} x' = 3x + y \\ y' = 4x \end{cases}$$

Solution. *Solving the first equation for y we get*

$$y = x' - 3x$$

and plugging it into the second equation we end up with:

$$x'' - 3x' = 4x$$

or rewritten:

$$x'' - 3x' - 4x = 0.$$

This equation has the general solution:

$$x(t) = c_1 e^{-t} + c_2 e^{4t}.$$

Plugging this into the first equaton in the system we have:

$$-c_1 e^{-t} + 4c_2 e^{4t} = 3c_1 e^{-t} + 3c_2 e^{4t} + y.$$

Now solving for y in the equation above we get:

$$y(t) = -4c_1 e^{-t} + c_2 e^{4t}.$$

So the solution to this system is the pair of equations:

$$\begin{aligned} x(t) &= c_1 e^{-t} + c_2 e^{4t} \\ y(t) &= -4c_1 e^{-t} + c_2 e^{4t} \end{aligned}$$

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Another example with all nonzero coefficients:

Example 2. *Solve the system of equation:*

$$\begin{cases} x' = -5x + y \\ y' = 2x + 5y \end{cases}$$

Solution. *Solving the first equation for* y *we get*

$$y = x' + 5x$$

and plugging it into the second equation we end up with:

$$x'' + 5x' = 2x + 5x' + 25x$$

or rewritten:

$$x'' - 27x = 0.$$

This equation has the general solution:

$$x(t) = c_1 e^{-\sqrt{27}t} + c_2 e^{\sqrt{27}t}.$$

Plugging this into the first equaton in the system we have:

$$-\sqrt{27}c_1 e^{-\sqrt{27}t} + \sqrt{27}c_2 e^{\sqrt{27}t} = -5c_1 e^{-\sqrt{27}t} - 5c_2 e^{\sqrt{27}t} + y.$$

Now solving for y *in the equation above we get:*

$$y(t) = (5 - \sqrt{27})c_1 e^{-\sqrt{27}t} + (5 + \sqrt{27})c_2 e^{\sqrt{27}t}.$$

So the solution to this system is the pair of equations:

$$\begin{aligned} x(t) &= c_1 e^{-\sqrt{27}t} + c_2 e^{\sqrt{27}t} \\ y(t) &= (5 - \sqrt{27})c_1 e^{-\sqrt{27}t} + (5 + \sqrt{27})c_2 e^{\sqrt{27}t} \end{aligned}$$

◇

7.2.2. *Linear Algebra Approach.*

Exercises. Solve the given system of equations using the Calculus method:

(1) $\begin{cases} x' = -5x \\ y' = -5x + 2y \end{cases}$

(2) $\begin{cases} x' = 7x \\ y' = -3x - 5y \end{cases}$

(3) $\begin{cases} x' = 6x + 5y \\ y' = 7x - 2y \end{cases}$

(4) $\begin{cases} x' = -6x + 5y \\ y' = 3x \end{cases}$

(5) $\begin{cases} x' = -6x + 2y \\ y' = -6x + 3y \end{cases}$

(6) $\begin{cases} x' = - 3y \\ y' = -2x + 2y \end{cases}$

7.3. Repeated Eigenvalues.

Exercises.

7.4. **The Two Body Problem.**

Exercises.

7.5. Additional Exercises.

8. Numerical Methods

8.1. Euler's Method.

Exercises.

8.2. **Taylor Series Method.**

Exercises.

8.3. **Runge-Kutta Method.**

Exercises.

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